Solution to Exercise 14.11
The parameters for the Leapfrog Theorem

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1 The constraints

We are given real numbers $t > 1$ and $0 < \alpha < 1$ (where α is the constant in the Sparse Ball Theorem). Consider real numbers θ, β, δ, μ, a, and h, and define

$$w := \frac{1}{2}\left(\cos\theta - \sin\theta - 1/t\right), \tag{1}$$

$$h' := h - \frac{2\beta(2+h)}{1-2\beta}, \tag{2}$$

$$a' := a - \frac{2\beta(a+\sin\theta)}{1-2\beta}, \tag{3}$$

$$\beta' := \frac{\beta(1+h)}{h+\cos\theta}, \tag{4}$$

$$\delta' := \frac{\delta(h+\cos\theta)}{1+h}, \tag{5}$$

and

$$\xi := \frac{12(1+h)(3a+\sin\theta)}{h' - 6a - 2\sin\theta - 8\beta(1+h) - 48\beta(1+h)^2}. \tag{6}$$

The parameters must be chosen, subject to the following constraints:

$$0 < \theta < \pi/4, \tag{7}$$

$$\cos\theta - \sin\theta > 1/t, \tag{8}$$

$$0 < \delta < 1, \tag{9}$$

$$\mu \geq \max\left(w, (1 + a + h/2 + \sin\theta)\left(1 + \frac{1}{\delta(1-2\beta)}\right)\right), \tag{10}$$

$$0 < \beta < \min\left(\delta, \cos\theta, \frac{h}{4(1+h)}, \frac{a}{4a+2\sin\theta}\right), \tag{11}$$

$$0 < a < h < \frac{\cos\theta - \beta}{1+\beta}, \tag{12}$$

$$h' > 6a + 2\sin\theta + 8\beta(1+h), \tag{13}$$

$$\beta(1+h)^2 < \delta(h+\cos\theta)^2, \tag{14}$$

$$\frac{\delta'(1-\beta')}{2(1-\beta') + \delta'(3-\beta')} \geq 1/6, \tag{15}$$

$$h' > 6a + 2\sin\theta + 8\beta(1+h) + 48\beta(1+h)^2, \tag{16}$$

$$\mu \geq (a+1+h/2)(1+1/\delta) + (\xi + \sin\theta)/\delta, \tag{17}$$

$$\xi < \cos\theta - h, \tag{18}$$

$$\cos\theta > 2(a+\sin\theta) + 1/t, \tag{19}$$

$$\begin{aligned} \xi \quad \leq \quad & \frac{(t\cos\theta - 1 - 2t(a+\sin\theta))\,(\cos\theta - h)}{t\cos\theta - 1 - 2t(a+\sin\theta) + 9t(1+h)} \\ & - \frac{3t(1+h)(2a+\sin\theta+2\beta(a+\sin\theta))}{t\cos\theta - 1 - 2t(a+\sin\theta) + 9t(1+h)}, \end{aligned} \tag{20}$$

$$\alpha a' - \sin\theta - \frac{6a\beta}{1-\beta} \geq \frac{1}{2}\,\alpha a, \tag{21}$$

and

$$\beta < \frac{a'}{2a+a'}. \tag{22}$$

2 Choice for the parameters

We define

$$\beta := \min\left(10^{-4}, \alpha/49\right), \tag{23}$$

$$\delta := 99/100, \tag{24}$$

$$h := 1/100, \tag{25}$$

$$\mu := 5, \tag{26}$$

$$a := \min\left(10^{-4}, \frac{t-1}{10^6 t}\right), \tag{27}$$

and choose θ, such that

$$0 < \theta < \pi/4, \tag{28}$$

$$\cos\theta > 91/100, \tag{29}$$

$$\sin\theta \leq \frac{t-1}{200t}, \tag{30}$$

$$\sin\theta \leq \alpha a/8, \tag{31}$$

and

$$\cos\theta \geq 1 + \frac{1375a}{9/1000 - 18a} - \frac{7}{9}\frac{t-1}{t}. \tag{32}$$

3 Verification of the constraints

In order for (32) to be possible, we need

$$0 < 1 + \frac{1375a}{9/1000 - 18a} - \frac{7}{9}\frac{t-1}{t} < 1.$$

Since, by (27), $a \leq 10^{-4}$, we have $9/1000 - 18a > 0$. Since the expression can be written as

$$\frac{1375a}{9/1000 - 18a} + \frac{2}{9} + \frac{7}{9t},$$

it follows that it is strictly positive. The requirement that it is less than one is equivalent to

$$\frac{1375a}{9/1000 - 18a} < \frac{7}{9}\frac{t-1}{t}.$$

Since, by (27), $a \leq 10^{-4}$, we have $9/1000 - 18a > 0$. Hence, we need

$$1375a < \frac{7}{9}\frac{t-1}{t}(9/1000 - 18a) = \frac{7}{1000}\frac{t-1}{t} - 14a\frac{t-1}{t},$$

which can be rewritten as

$$\left(1375 + 14\frac{t-1}{t}\right) a < \frac{7}{1000}\frac{t-1}{t}.$$

Since $1375 + 14(t-1)/t \leq 1389$, it is sufficient to show that

$$1389a < \frac{7}{1000}\frac{t-1}{t},$$

i.e.,

$$a < \frac{7}{1000 \cdot 1389}\frac{t-1}{t}.$$

The latter inequality holds, because, using (27),

$$a \leq \frac{1}{10^6}\frac{t-1}{t} < \frac{7}{1000 \cdot 1389}\frac{t-1}{t}.$$

3.1 Verification of (7)

Constraint (7) holds, because of (28).

3.2 Verification of (8)

Using (30) and (32), we have

$$\cos\theta - \sin\theta > 1 - \frac{7}{9}\frac{t-1}{t} - \frac{t-1}{200t} = \left(\frac{2}{9} - \frac{1}{200}\right) + \left(\frac{7}{9} + \frac{1}{200}\right)\frac{1}{t}.$$

Thus, to show that constraint (8) holds, it suffices to show that

$$\left(\frac{2}{9} - \frac{1}{200}\right) + \left(\frac{7}{9} + \frac{1}{200}\right)\frac{1}{t} > \frac{1}{t},$$

which is equivalent to

$$\left(\frac{2}{9} - \frac{1}{200}\right) > \left(\frac{2}{9} - \frac{1}{200}\right)\frac{1}{t},$$

which holds, because $t > 1$.

3.3 Verification of (9)

Constraint (9) holds, because of (24).

3.4 Verification of (10)

Using (1) and (26), we have

$$w = \frac{1}{2}\left(\cos\theta - \sin\theta - 1/t\right) \leq 1/2 \leq 5 = \mu.$$

Using (23), (24), (25), and (27), we have

$$\begin{aligned}(1 + a + h/2 + \sin\theta)\left(1 + \frac{1}{\delta(1-2\beta)}\right) &\leq \left(1 + \frac{1}{10^4} + \frac{1}{200} + 1\right)\left(1 + \frac{1}{\frac{99}{100}(1 - 2\cdot 10^{-4})}\right) \\ &\leq 5 \\ &= \mu.\end{aligned}$$

3.5 Verification of (11)

Using (23) and (24), we have $0 < \beta < \delta$. Using (23) and (29), we have

$$\cos\theta > 91/100 > 10^{-4} \geq \beta.$$

Using (23) and (25), we have

$$\frac{h}{4(1+h)} = 1/404 > 10^{-4} \geq \beta.$$

The inequality

$$\beta < \frac{a}{4a + 2\sin\theta}$$

is equivalent to

$$(4a + 2\sin\theta)\beta < a.$$

Using (31) and the fact that $0 < \alpha < 1$, we have

$$(4a + 2\sin\theta)\beta \leq (4a + \alpha a/4)\beta < 5a\beta,$$

which is less than a, because, by (23), $\beta < 1/5$.

3.6 Verification of (12)

Using (25) and (27), we have $0 < a \leq 10^{-4} < h$. The inequality

$$h < \frac{\cos\theta - \beta}{1 + \beta}$$

is equivalent to

$$\cos\theta > \beta + h(1 + \beta).$$

Using (23), (25), and (29), we have

$$\beta + h(1 + \beta) \leq 10^{-4} + \frac{1}{100}\left(1 + 10^{-4}\right) < 91/100 < \cos\theta.$$

3.7 Verification of (13)

The constraint (13) is implied by (16), which will be verified later.

3.8 Verification of (14)

Using (23) and (25), we have

$$\beta(1 + h)^2 \leq \frac{1}{10^4}\left(1 + \frac{1}{100}\right)^2 = \frac{101^2}{10^8}.$$

Thus, it suffices to show that

$$\frac{101^2}{10^8} < \delta(h + \cos\theta)^2.$$

Using (24) and (25), this is equivalent to

$$\frac{101^2}{10^8} < \frac{99}{100}\left(\frac{1}{100} + \cos\theta\right)^2,$$

which is equivalent to

$$\left(\frac{1}{100} + \cos\theta\right)^2 > \frac{101^2}{99 \cdot 10^6}.$$

Using (29), we have

$$\left(\frac{1}{100} + \cos\theta\right)^2 > \left(\frac{92}{100}\right)^2 > \frac{101^2}{99 \cdot 10^6}.$$

3.9 Verification of (15)

We first show that

$$\frac{9}{10}\delta < \delta' < \delta. \tag{33}$$

The inequality $\delta' < \delta$ follows from the definition of δ' in (5). The inequality $\frac{9}{10}\delta < \delta'$ is equivalent to

$$\frac{9}{10} < \frac{h + \cos\theta}{1 + h},$$

which is equivalent to

$$10\cos\theta + h > 9.$$

Using (29), we have

$$10\cos\theta + h > 10\cos\theta > \frac{91}{10} > 9.$$

This proves that (33) holds.

In exactly the same way, using the definition of β' in (4), we get

$$\frac{9}{10}\beta' < \beta < \beta'. \tag{34}$$

Now we can show that constraint (15) is satisfied. Using (33) and (34), we get

$$\frac{\delta'(1-\beta')}{2(1-\beta') + \delta'(3-\beta')} \geq \frac{\frac{9}{10}\delta\left(1 - \frac{10}{9}\beta\right)}{2 + 3\delta},$$

which, using (23) and (24), is at least

$$\frac{\frac{9}{10}\frac{99}{100}\left(1 - \frac{10}{9}\cdot 10^{-4}\right)}{2 + \frac{3\cdot 99}{100}} \geq 1/6.$$

3.10 Verification of (16)

Using the definition of h' in (2), the constraint (16) can be written as

$$h > \frac{2\beta(2+h)}{1-2\beta} + 6a + 2\sin\theta + 8\beta(1+h) + 48\beta(1+h)^2.$$

Using (23), (25), (27), and (31), and the fact that $0 < \alpha < 1$, we get

$$\begin{aligned} &\frac{2\beta(2+h)}{1-2\beta} + 6a + 2\sin\theta + 8\beta(1+h) + 48\beta(1+h)^2 \\ &< \frac{\frac{2}{10^4}(2 + 1/100)}{1 - \frac{2}{10^4}} + \frac{6}{10^4} + \frac{1}{4\cdot 10^4} + \frac{8}{10^4}\left(1 + \frac{1}{100}\right) + \frac{48}{10^4}\left(1 + \frac{1}{100}\right)^2, \end{aligned}$$

which is less than $1/100 = h$.

3.11 Some inequalities for ξ

The value of ξ is defined in (6). We first prove that

$$\xi \leq \frac{25a}{\frac{2}{1000} - 4a}. \tag{35}$$

Using the definitions of h' and ξ in (6) and (2), respectively, we have

$$\xi = \frac{12(1+h)(3a+\sin\theta)}{h - \frac{2\beta(2+h)}{1-2\beta} - 6a - 2\sin\theta - 8\beta(1+h) - 48\beta(1+h)^2}.$$

Using (23), (25), (27), and (31), and the fact that $0 < \alpha < 1$, we have

$$\begin{aligned} \xi &\leq \frac{12 \cdot \frac{101}{100}(3a + a/8)}{\frac{1}{100} - \frac{2\cdot 10^{-4}.\frac{201}{100}}{1-2\cdot 10^{-4}} - 6a - \frac{1}{4\cdot 10^4} - \frac{8}{10^4}\cdot\frac{101}{100} - \frac{48}{10^4}\left(\frac{101}{100}\right)^2} \\ &= \frac{\frac{75}{2}a}{\frac{1}{101} - \frac{1}{10^4}\left(\frac{402}{101(1-2\cdot 10^{-4})} + \frac{100}{404} + 8 + \frac{48\cdot 101}{100}\right) - \frac{600}{101}a}. \end{aligned}$$

Since

$$\frac{1}{101} - \frac{1}{10^4}\left(\frac{402}{101\,(1-2\cdot 10^{-4})} + \frac{100}{404} + 8 + \frac{48\cdot 101}{100}\right) > 3/1000,$$

and since

$$\frac{600}{101}a \leq 6a,$$

we have

$$\xi \leq \frac{\frac{75}{2}a}{\frac{3}{1000} - 6a} = \frac{25a}{\frac{2}{1000} - 4a},$$

completing the proof of (35).

We next prove that

$$\xi \leq \frac{9}{110}(\cos\theta - 1/t) - \frac{1}{55}\frac{t-1}{t}. \tag{36}$$

To prove this inequality, we observe that, using (35), it suffices to show that

$$\frac{25a}{\frac{2}{1000} - 4a} \leq \frac{9}{110}(\cos\theta - 1/t) - \frac{1}{55}\frac{t-1}{t},$$

which is equivalent to

$$\cos\theta \geq \frac{110}{9}\frac{25a}{\frac{2}{1000} - 4a} + \frac{1}{t} + \frac{110}{9}\frac{1}{55}\frac{t-1}{t} = 1 + \frac{1375a}{9/1000 - 18a} - \frac{7}{9}\frac{t-1}{t},$$

which is true, because of (32).

3.12 Verification of (17)

Using (27) and (35), we get

$$\xi \leq \frac{25a}{\frac{2}{1000} - 4a} \leq \frac{25 \cdot 10^{-4}}{\frac{2}{1000} - 4 \cdot 10^{-4}} = \frac{25}{16}.$$

Thus, using (24), (25), and (27), we get

$$\begin{aligned}(a+1+h/2)(1+1/\delta) + (\xi + \sin\theta)/\delta &\leq \left(\frac{1}{10^4} + 1 + \frac{1}{200}\right)\left(1 + \frac{100}{99}\right) + \frac{\frac{25}{16}+1}{99/100} \\ &= \left(\frac{1}{10^4} + 1 + \frac{1}{200}\right)\frac{199}{99} + \frac{100 \cdot 41}{99 \cdot 16} \\ &< 5 \\ &= \mu.\end{aligned}$$

3.13 Verification of (18)

It follows from (36) that

$$\xi \leq \frac{9}{110}.$$

Using (25) and (29), we get

$$\xi + h \leq \frac{9}{110} + \frac{1}{100} < \frac{91}{100} < \cos\theta.$$

3.14 Verification of (19)

Using (27) and (30), we get

$$\begin{aligned}2(a + \sin\theta) + \frac{1}{t} &\leq \frac{2}{10^6}\frac{t-1}{t} + \frac{1}{100}\frac{t-1}{t} + \frac{1}{t} \\ &\leq \frac{2}{9}\frac{t-1}{t} + \frac{1}{t} \\ &= 1 - \frac{7}{9}\frac{t-1}{t},\end{aligned}$$

which, by (32), is less than $\cos\theta$.

3.15 Verification of (20)

The constraint (20) is equivalent to

$$\begin{aligned}\xi\left(t\cos\theta - 1 - 2t(a+\sin\theta) + 9t(1+h)\right) + 3t(1+h)(2a + \sin\theta + 2\beta(a + \sin\theta)) \\ \leq \left(t\cos\theta - 1 - 2t(a+\sin\theta)\right)(\cos\theta - h). \qquad (37)\end{aligned}$$

Observe that

$$t\cos\theta - 1 - 2t(a+\sin\theta) \le t.$$

Using (23), (25), (27), and (30), we get

$$\begin{aligned} 2a+\sin\theta+2\beta(a+\sin\theta) &\le \frac{2}{10^6}\frac{t-1}{t}+\frac{1}{200}\frac{t-1}{t}+\frac{2}{10^4}\left(\frac{1}{10^6}\frac{t-1}{t}+\frac{1}{200}\frac{t-1}{t}\right) \\ &\le \frac{1}{30(1+h)}\frac{t-1}{t}. \end{aligned}$$

Using (25) and (29), we get

$$\cos\theta - h \ge \frac{91}{100} - \frac{1}{100} = \frac{9}{10}.$$

Hence, (37) will follow from the claim that

$$\xi\left(t+9t(1+h)\right)+\frac{1}{10}(t-1) \le \frac{9}{10}\left(t\cos\theta - 1 - 2t(a+\sin\theta)\right).$$

Since, using (25), $t+9t(1+h) \le 11t$, it suffices to show that

$$11\xi t + \frac{1}{10}(t-1) \le \frac{9}{10}\left(t\cos\theta - 1 - 2t(a+\sin\theta)\right),$$

which is equivalent to

$$11\xi t + \frac{1}{10}(t-1) + \frac{18}{10}t(a+\sin\theta) \le \frac{9}{10}\left(t\cos\theta - 1\right).$$

Using (27) and (30), we get

$$\frac{18}{10}t(a+\sin\theta) \le \frac{18}{10}t\left(\frac{1}{10^6}\frac{t-1}{t}+\frac{1}{200}\frac{t-1}{t}\right) = \left(\frac{18}{10^7}+\frac{9}{1000}\right)(t-1) \le \frac{1}{10}(t-1).$$

Thus, it suffices to show that

$$11\xi t + \frac{1}{5}(t-1) \le \frac{9}{10}\left(t\cos\theta - 1\right),$$

which is equivalent to

$$\xi \le \frac{9}{110}\left(\cos\theta - 1/t\right) - \frac{1}{55}\frac{t-1}{t},$$

which we have shown to hold in (36).

3.16 Verification of (21)

We have to show that

$$\frac{1}{2}\,\alpha a + \sin\theta + \frac{6a\beta}{1-\beta} \leq \alpha a'.$$

We know from (31) that

$$\sin\theta \leq \alpha a/8.$$

Using (23) and the fact that $0 < \alpha < 1$, we get

$$48\beta + \alpha\beta \leq 49\beta \leq \alpha,$$

which can be rewritten as $48\beta \leq \alpha(1-\beta)$, which can be rewritten as

$$\frac{\beta}{1-\beta} \leq \alpha/48.$$

Thus, it suffices to show that

$$\frac{1}{2}\,\alpha a + \frac{1}{8}\,\alpha a + \frac{1}{8}\,\alpha a \leq \alpha a',$$

which is equivalent to

$$\frac{3}{4}a \leq a'. \tag{38}$$

Using the definition of a' in (3), this is equivalent to

$$a - \frac{2\beta(a+\sin\theta)}{1-2\beta} \geq \frac{3}{4}a,$$

which is equivalent to

$$\frac{2\beta(a+\sin\theta)}{1-2\beta} \leq \frac{1}{4}a.$$

Using (31) and the fact that $0 < \alpha < 1$, we have

$$\sin\theta \leq \alpha a/8 \leq a/8.$$

Thus, it suffices to show that

$$\frac{2\beta(a+a/8)}{1-2\beta} \leq \frac{1}{4}a,$$

which is equivalent to

$$\frac{\frac{9}{4}\beta}{1-2\beta} \leq \frac{1}{4},$$

which is equivalent to

$$9\beta \leq 1-2\beta,$$

which is equivalent to

$$\beta \leq 1/11,$$

which is true, because of (23).

3.17 Verification of (22)

We have to show that

$$\beta < \frac{a'}{2a + a'},$$

which is equivalent to

$$(2a + a')\beta < a'.$$

We have seen in (38) that $3a/4 \leq a'$. Also, the definition of a' in (3) implies that $a' < a$. Therefore, it suffices to show that

$$(2a + a)\beta < \frac{3}{4}a,$$

which is equivalent to

$$\beta < 1/4,$$

which is true, because of (23).