# COT 5407: Introduction to Algorithms

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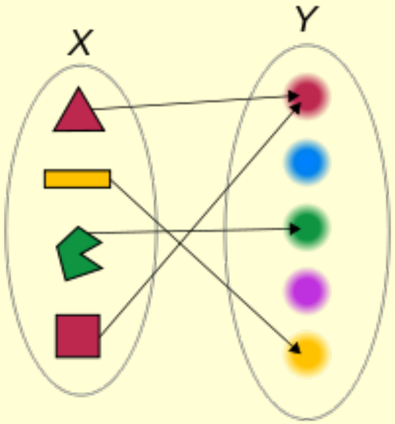
# Sorting Algorithms

- SelectionSort
- InsertionSort
- BubbleSort
- ShakerSort
- MergeSort
- HeapSort
- QuickSort
- Bucket & Radix Sort
- Counting Sort

# Definitions

**Abstract Problem**: defines a function from any allowable input to a corresponding output




**Instance of a Problem**: a specific input to abstract problem

**Algorithm**: well-defined computational procedure that takes an instance of a problem as input and produces the correct output

**An Algorithm must <u>halt</u> on every input with <u>correct</u> output.**

# Algorithm Analysis

- Worst-case time complexity*
- (Worst-case) space complexity
- Average-case time complexity

# Worst-Case Analysis

Two Techniques:

1. Counts and Summations:
   - Count number of steps from pseudocode and add
2. Recurrence Relations:
   - Use invariant, write down recurrence relation and solve it

We will use big-Oh notation to write down time and space complexity (for both worst-case & average-case analyses).

Compute the worst possible time of all input instances of length N.

# Definition of big-Oh

- We say that
  - $F(n) = O(G(n))$

  If there exists two <u>positive</u> constants, c and $n_0$, such that
  - For all $n \geq n_0$, we have $F(n) \leq c\, G(n)$
- Thus, to show that $F(n) = O(G(n))$, you need to find two positive constants that satisfy the condition mentioned above
- Also, to show that $F(n) \neq O(G(n))$, you need to show that for any value of c, there does not exist a positive constant $n_0$ that satisfies the condition mentioned above

# SelectionSort – Worst-case analysis

$\textsc{SelectionSort}(array\ A)$

$N \leftarrow length[A]$
**for** $p \leftarrow 1$ **to** $N$
    **do** $\triangleright$ Compute $j$
        $j \leftarrow p$
        **for** $m \leftarrow p + 1$ **to** $N$
            **do if** $\boxed{(A[m] < A[j])}$
                **then** $j \leftarrow m$
        $\triangleright$ Swap $A[p]$ and $A[j]$
        $temp \leftarrow A[p]$
        $A[p] \leftarrow A[j]$
        $A[j] \leftarrow temp$

N-p comparisons

3 data movements

# SelectionSort: Worst-Case Analysis

- Data Movements

$$= \sum_{p=1}^{N} 3 = 3 \times N = O(N)$$

- Number of Comparisons

$$\begin{aligned} &= \sum_{p=1}^{N} (N - p) \\ &= \sum_{p=1}^{N} N - \sum_{p=1}^{N} p \\ &= (N \times N) - (N)(N+1)/2 \\ &= O(N^2) \end{aligned}$$

Learn how to sum series

- Time Complexity = $O(N^2)$

# SelectionSort – Worst-case space analysis

$\textsc{SelectionSort}(array\ A)$

$N \leftarrow length[A]$

**for** $p \leftarrow 1$ **to** $N$

  **do** $\triangleright$ Compute $j$

    $\boxed{j \leftarrow p}$

    **for** $m \leftarrow p + 1$ **to** $N$

      **do if** $(A[m] < A[j])$

        **then** $j \leftarrow m$

    $\triangleright$ Swap $A[p]$ and $A[j]$

    $\boxed{temp \leftarrow A[p]}$

    $A[p] \leftarrow A[j]$

    $A[j] \leftarrow temp$

- Temp Space
  - No extra arrays or data structures
  - $O(1)$

# MergeSort

- Divide-and-Conquer Strategy
- Divide array into two sublists of roughly equal length
- Sort each sublist "recursively"
- Merge two sorted lists to get final sorted list
  - **Assumption**: Merging is faster than sorting from fresh
- Most of the work is done in merging
- Process described using a tree
  - **Top-down process**: Divide each list into 2 sublists
  - **Bottom-up process**: Merge two sorted sublists into one sorted sublist

# MergeSort

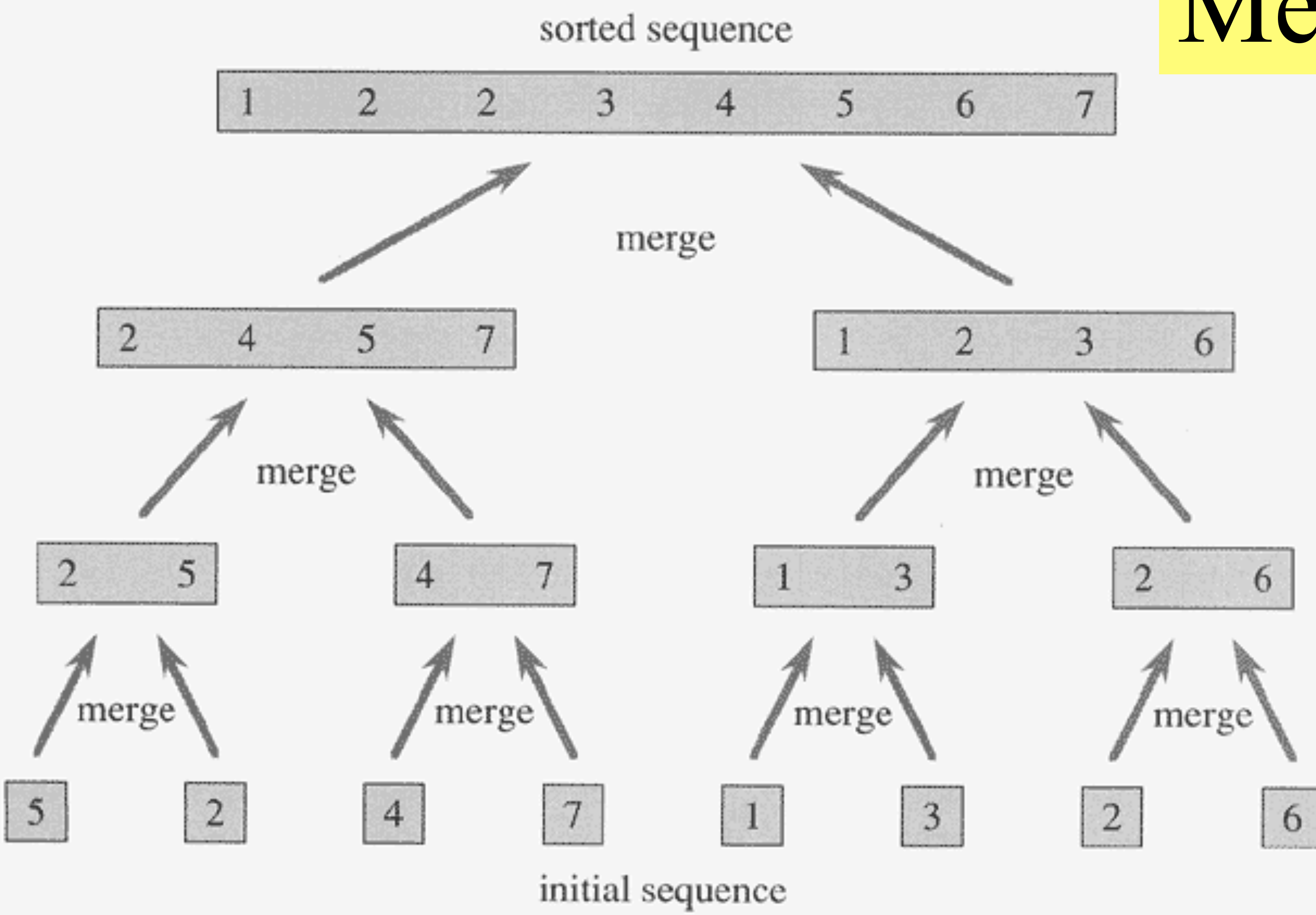
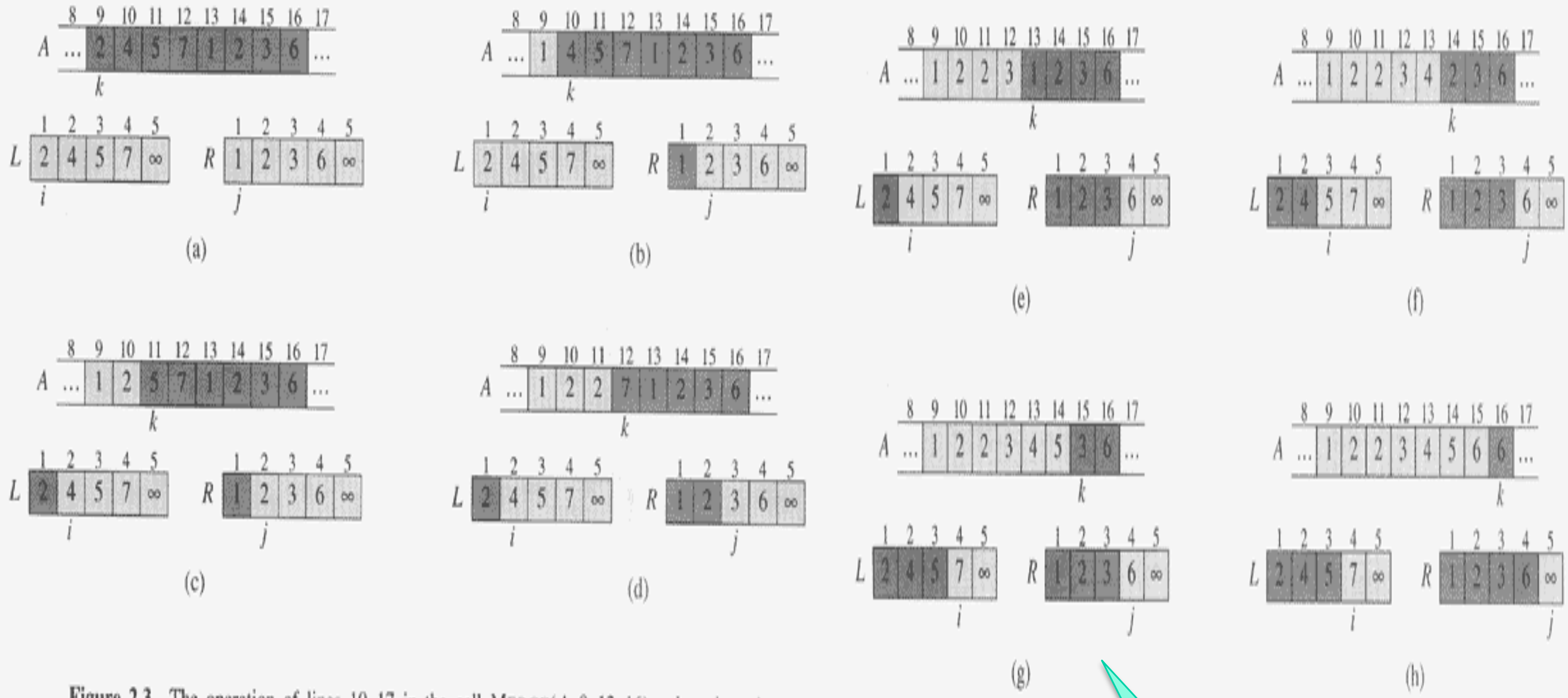



**Figure 2.4** The operation of merge sort on the array $A = \langle 5, 2, 4, 7, 1, 3, 2, 6 \rangle$. The lengths of the sorted sequences being merged increase as the algorithm progresses from bottom to top.



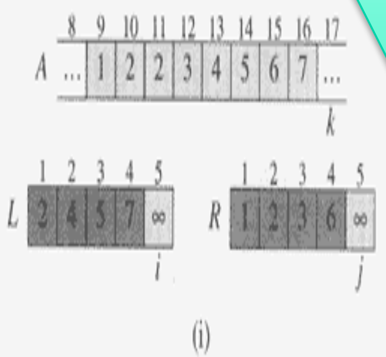


**Figure 2.3** The operation of lines 10–17 in the call MERGE($A$, 9, 12, 16), when the subarray $A[9..16]$ contains the sequence $\langle 2, 4, 5, 7, 1, 2, 3, 6 \rangle$. After copying and inserting sentinels, the array $L$ contains $\langle 2, 4, 5, 7, \infty \rangle$, and the array $R$ contains $\langle 1, 2, 3, 6, \infty \rangle$. Lightly shaded positions in $A$ contain their final values, and lightly shaded positions in $L$ and $R$ contain values that have yet to be copied back into $A$. Taken together, the lightly shaded positions always comprise the values originally in $A[9..16]$, along with the two sentinels. Heavily shaded positions in $A$ contain values that will be copied over, and heavily shaded positions in $L$ and $R$ contain values that have already been copied back into $A$. **(a)–(h)** The arrays $A$, $L$, and $R$, and their respective indices $k$, $i$, and $j$ prior to each iteration of the loop of lines 12–17. **(i)** The arrays and indices at termination. At this point, the subarray in $A[9..16]$ is sorted, and the two sentinels in $L$ and $R$ are the only two elements in these arrays that have not been copied into $A$.

Merge uses an extra array & lots of data movements

MERGE$(A, p, q, r)$

$n_1 \leftarrow q - p + 1$
$n_2 \leftarrow r - q$
create arrays $L[1 \, .. \, n_1 + 1]$ and $R[1 \, .. \, n_2 + 1]$
**for** $i \leftarrow 1$ **to** $n_1$
    **do** $L[i] \leftarrow A[p + i - 1]$
**for** $j \leftarrow 1$ **to** $n_2$
    **do** $R[j] \leftarrow A[q + j]$
$L[n_1 + 1] \leftarrow \infty$
$R[n_2 + 1] \leftarrow \infty$
$i \leftarrow 1$
$j \leftarrow 1$
**for** $k \leftarrow p$ **to** $r$
    **do if** $L[i] \leq R[j]$
        **then** $A[k] \leftarrow L[i]$
            $i \leftarrow i + 1$
        **else** $A[k] \leftarrow R[j]$
            $j \leftarrow j + 1$

**Assumption**: Array A is sorted from [p..q] and from [q+1..r].

**Space**: Two extra arrays L and R are used.

**Sentinel Items**: Two sentinel items placed in lists L and R.

**Merge**: The smaller of the item in L and item in R is moved to next location in A

**Time** : O(length of lists)

# MergeSort

```
MERGE-SORT(A, p, r)
  if p < r
     then q ← ⌊(p + r)/2⌋
          MERGE-SORT(A, p, q)
          MERGE-SORT(A, q + 1, r)
          MERGE(A, p, q, r)
```

Time Complexity Recurrence: $T(N) = 2T(N/2) + O(N)$

# Solving Recurrence Relations

| Recurrence; Cond | Solution |
|---|---|
| $T(n) = T(n-1) + O(1)$ | $T(n) = O(n)$ |
| $T(n) = T(n-1) + O(n)$ | $T(n) = O(n^2)$ |
| $T(n) = T(n-c) + O(1)$ | $T(n) = O(n)$ |
| $T(n) = T(n-c) + O(n)$ | $T(n) = O(n^2)$ |
| $T(n) = 2T(n/2) + O(n)$ | $T(n) = O(n \log n)$ |
| $T(n) = aT(n/b) + O(n)$; $a = b$ | $T(n) = O(n \log n)$ |
| $T(n) = aT(n/b) + O(n)$; $a < b$ | $T(n) = O(n)$ |
| $T(n) = aT(n/b) + f(n)$; $f(n) = O(n^{\log_b a - \epsilon})$ | $T(n) = O(n)$ |
| $T(n) = aT(n/b) + f(n)$; $f(n) = O(n^{\log_b a})$ | $T(n) = \Theta(n^{\log_b a} \log n)$ |
| $T(n) = aT(n/b) + f(n)$; $f(n) = \Theta(f(n))$ $af(n/b) \leq cf(n)$ | $T(n) = \Omega(n^{\log_b a} \log n)$ |

# Solving Recurrences: Recursion-tree method

- Substitution method fails when a good guess is not available
- Recursion-tree method works in those cases
  - Write down the recurrence as a tree with recursive calls as the children
  - Expand the children
  - Add up each level
  - Sum up the levels
- Useful for analyzing divide-and-conquer algorithms
- Also useful for generating good guesses to be used by substitution method

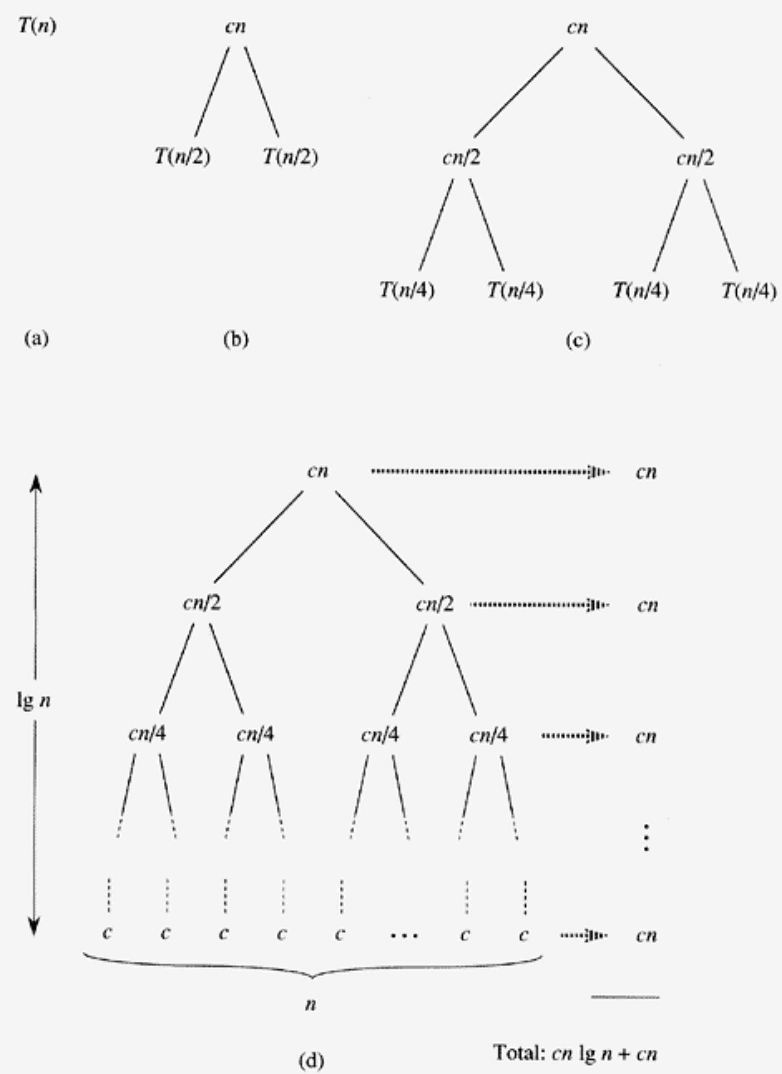



**Figure 2.5** The construction of a recursion tree for the recurrence $T(n) = 2T(n/2) + cn$. Part **(a)** shows $T(n)$, which is progressively expanded in **(b)–(d)** to form the recursion tree. The fully expanded tree in part (d) has $\lg n + 1$ levels (i.e., it has height $\lg n$, as indicated), and each level contributes a total cost of $cn$. The total cost, therefore, is $cn \lg n + cn$, which is $\Theta(n \lg n)$.

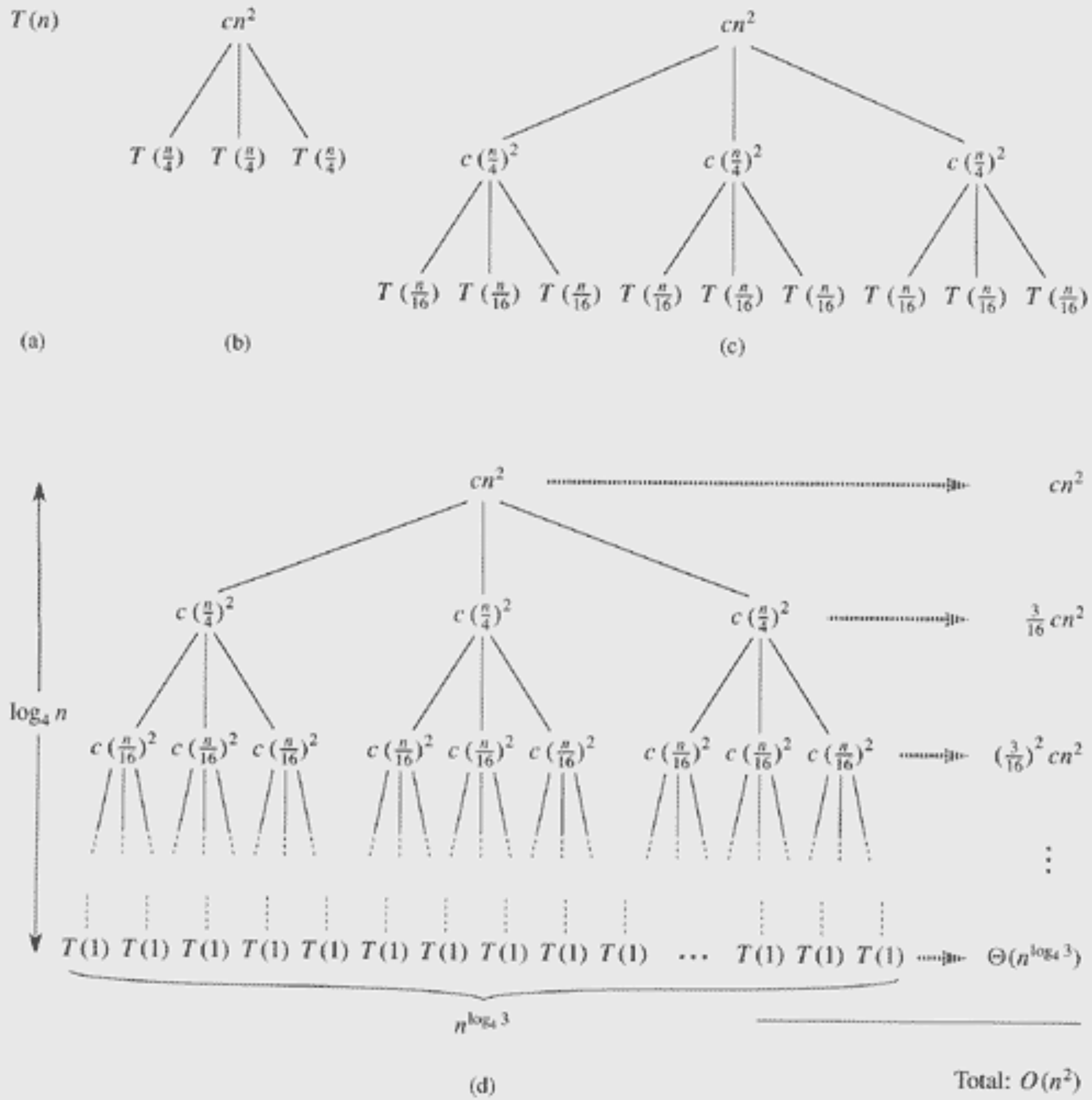



**Figure 4.1** The construction of a recursion tree for the recurrence $T(n) = 3T(n/4) + cn^2$. Part **(a)** shows $T(n)$, which is progressively expanded in **(b)–(d)** to form the recursion tree. The fully expanded tree in part (d) has height $\log_4 n$ (it has $\log_4 n + 1$ levels).

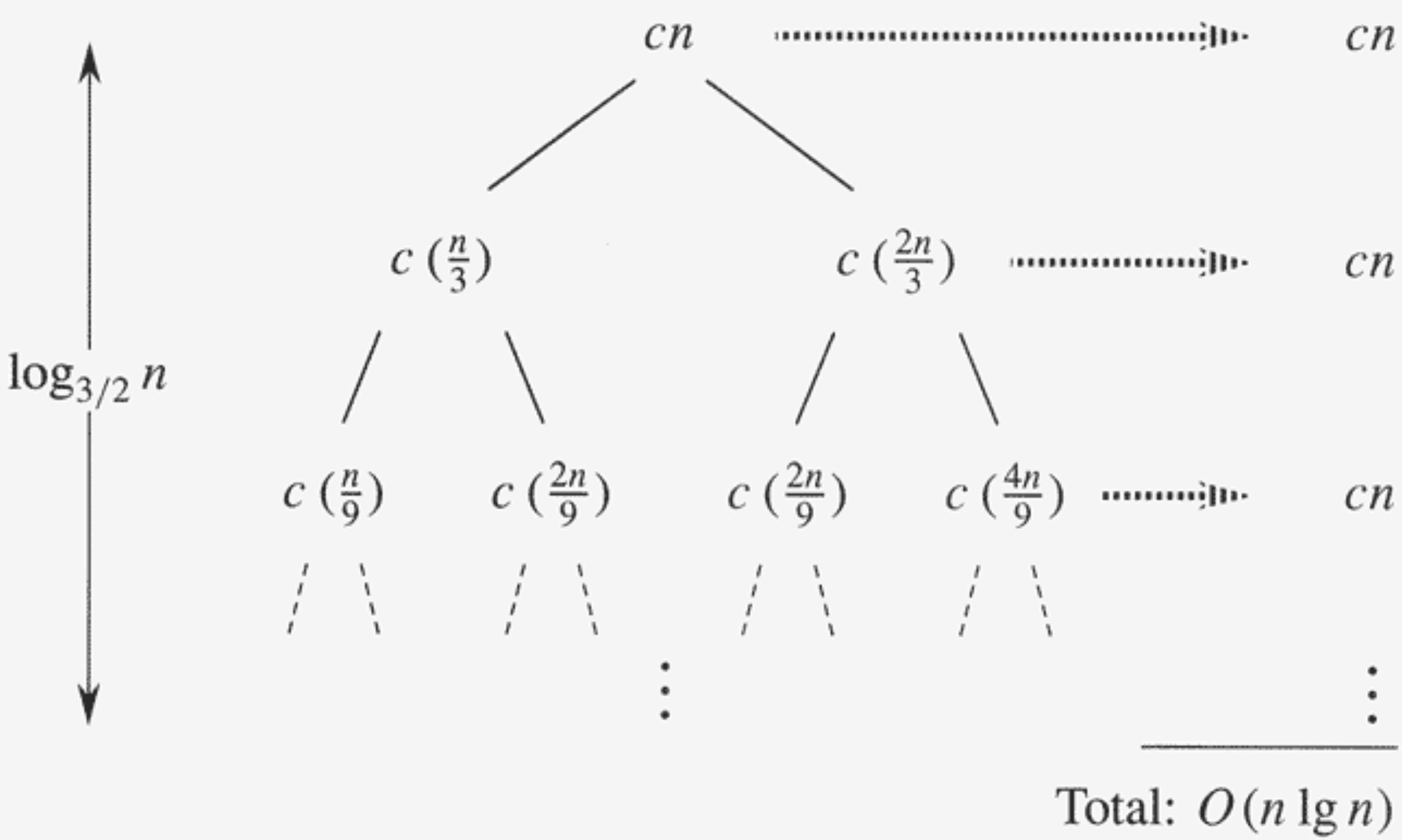



**Figure 4.2** A recursion tree for the recurrence $T(n) = T(n/3) + T(2n/3) + cn$.

# Solving Recurrences using Master Theorem

**Master Theorem**:

Let a,b >= 1 be constants, let f(n) be a function, and let

$$T(n) = aT(n/b) + f(n)$$

1. If $f(n) = O(n^{\log_b a-e})$ for some constant e>0, then
   $T(n) = \Theta(n^{\log_b a})$
2. If $f(n) = \Theta(n^{\log_b a})$, then
   $T(n) = \Theta(n^{\log_b a} \log n)$
3. If $f(n) = \Omega(n^{\log_b a+e})$ for some constant e>0, then
   $T(n) = \Theta(f(n))$

# Solving Recurrences by Substitution

- Guess the form of the solution
- (Using mathematical induction) find the constants and show that the solution works

**Example**

$$T(n) = 2T(n/2) + n$$

| | | |
|---|---|---|
| Guess (#1) | $T(n) = O(n)$ | |
| Need | $T(n) \le cn$ | for some constant $c>0$ |
| Assume | $T(n/2) \le cn/2$ | Inductive hypothesis |
| Thus | $T(n) \le 2cn/2 + n = (c+1)\,n$ | |
| | **Our guess was wrong!!** | |

# Solving Recurrences by Substitution: 2

$$T(n) = 2T(n/2) + n$$

| | | |
|---|---|---|
| Guess (#2) | $T(n) = O(n^2)$ | |
| Need | $T(n) <= cn^2$ | for some constant $c>0$ |
| Assume | $T(n/2) <= cn^2/4$ | Inductive hypothesis |
| Thus | $T(n) <= 2cn^2/4 + n = cn^2/2+ n$ | |

**Works for all n as long as c>=2 !!**

**But there is a lot of "slack"**

# Solving Recurrences by Substitution: 3

## $T(n) = 2T(n/2) + n$

Guess (#3) $T(n) = O(n\log n)$

Need **$T(n) <= cn\log n$** for some constant $c>0$

Assume $T(n/2) <= c(n/2)(\log(n/2))$ Inductive hypothesis

Thus $T(n) <= 2\ c(n/2)(\log(n/2)) + n$

$<= cn\log n - cn + n <= cn\log n$

**Works for all n as long as c>=1 !!**

**This is the correct guess. WHY?**

Show $T(n) >= c' n\log n$ for some constant $c' >0$

# Solving Recurrence Relations

| Recurrence; Cond | Solution |
|---|---|
| $T(n) = T(n-1) + O(1)$ | $T(n) = O(n)$ |
| $T(n) = T(n-1) + O(n)$ | $T(n) = O(n^2)$ |
| $T(n) = T(n-c) + O(1)$ | $T(n) = O(n)$ |
| $T(n) = T(n-c) + O(n)$ | $T(n) = O(n^2)$ |
| $T(n) = 2T(n/2) + O(n)$ | $T(n) = O(n \log n)$ |
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| $T(n) = aT(n/b) + O(n)$; $a < b$ | $T(n) = O(n)$ |
| $T(n) = aT(n/b) + f(n)$; $f(n) = O(n^{\log_b a - \epsilon})$ | $T(n) = O(n)$ |
| $T(n) = aT(n/b) + f(n)$; $f(n) = O(n^{\log_b a})$ | $T(n) = \Theta(n^{\log_b a} \log n)$ |
| $T(n) = aT(n/b) + f(n)$; $f(n) = \Theta(f(n))$ $af(n/b) \leq cf(n)$ | $T(n) = \Omega(n^{\log_b a} \log n)$ |

# QuickSort

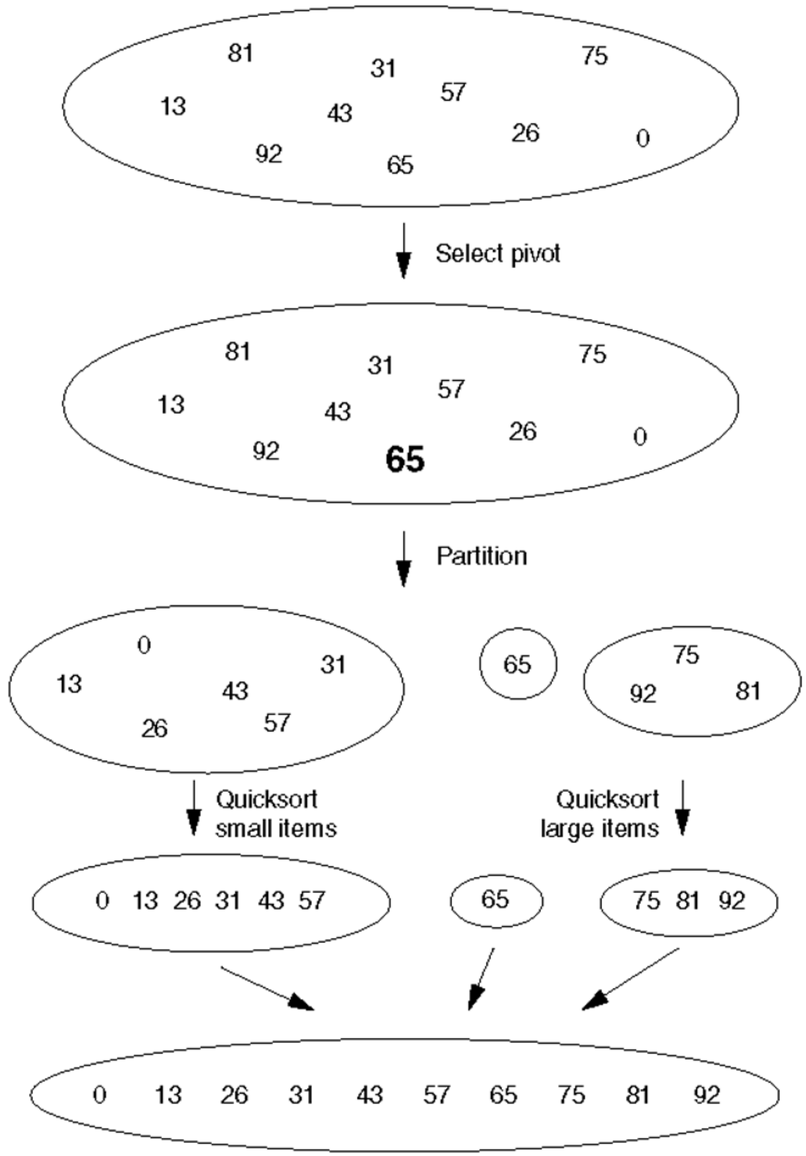
MergeSort

- Divide into 2 equal sublists
- Sort each sublist "recursively"
- Merge 2 sorted sublists
  - **Assumption**: Merging is faster than sorting from fresh
- Most of work is done in merging

QuickSort

- Partition into 2 sublists using a pivot
- Sort each sublist "recursively"
- Concatenate 2 sorted sublists
  - **Assumption**: Partition is faster than sorting
- Most of work is done in partition
- Process described using a tree
  - **Top-down process**: Partition each list into 2 sublists
  - **Bottom-up process**: Concatenate two sorted sublists into one sorted sublist

**Figure 8.10** Quicksort



# Partition Algorithm

- Pick a pivot
- Compare each item to a pivot and create two lists:
  - L = list of all items smaller than the pivot
  - R = list of all items larger than the pivot

O(N) time

- One scan through the list is enough, but seems to need extra space
- How to design an **in-place partition algorithm**!

# Partition

**Figure A** If 6 is used as pivot, the end result after partitioning is as shown in the Figure B.

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 4 | 5 | 0 | 3 | 9 | 8 | 7 | 6 |

**Figure B** Result after Partitioning

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 1 | 4 | 5 | 0 | 3 | 6 | 8 | 7 | 9 |

## QuickSort

```
QuickSort(array A, int p, int r)
if (p < r)
    then q ← Partition(A, p, r)
         QuickSort(A, p, q − 1)
         QuickSort(A, q + 1, r)
```

To sort array call QuickSort($A$, 1, *length*[$A$]).

```
Partition(array A, int p, int r)
x ← A[r]                          ▷ Choose pivot
i ← p − 1
for j ← p to r − 1
      do if (A[j] ≤ x)
            then i ← i + 1
                 exchange A[i] ↔ A[j]
exchange A[i + 1] ↔ A[r]
return i + 1
```

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# Time Complexity

- $T(N) = O(N) + T(N_1) + T(N_2)$
- On the average, $N_1 = N_2 = N/2$
- Thus, average-case complexity = $O(N \log N)$
- Worst-case: Either $N_1$ or $N_2 = 0$
  - Thus, $T(N) = O(N) + T(N - 1)$
  - $T(N) = O(N^2)$