Efficient algorithms for constructing fault-tolerant geometric spanners

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Abstract

Let S be a set of n points in $\mathbb{R}^d$, and k an integer such that $1 \leq k \leq n-2$. Algorithms are given that construct fault-tolerant spanners for S. If in such a spanner at most k edges or vertices are removed, then each pair of points in the remaining graph is still connected by a short path. Our results include (i) an algorithm with running time $O(n \log^{d-1} n + kn \log\log n + k^2 n)$ that constructs a spanner with $O(k^2 n)$ edges, that is resilient to k edge faults, (ii) an algorithm with running time $O(n \log n + k^2 n)$ that constructs a spanner with $O(k^2 n)$ edges, that is resilient to k vertex faults, and (iii) an algorithm with running time $O(n \log n + c^k n)$ that constructs a spanner of degree $O(c^k)$, whose total edge length is bounded by $O(c^k)$ times the weight of a minimum spanning tree of S, and that is resilient to k edge or vertex faults. Here, c is a constant that is independent of n and k.

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1 Introduction

Spanners have applications in the design of geometric networks. Consider a set S of n points in $\mathbb{R}^d$, where the dimension d is a constant. A geometric network on S can be modeled as an undirected graph G with vertex set S and with edges $e = (a, b)$ of length $|e|$ that is defined as the Euclidean distance $|ab|$ between its two endpoints a and b. Let p and q be two points of S, and let P be a pq-path in G, i.e., a path in G between p and q. The length $|P|$ of P is defined as the sum of the lengths of the edges of P. Let $t > 1$ be a real number. We say that G is a *t-spanner* for S, if for each pair of points $p, q \in S$, there exists a pq-path in G of length at most t times the Euclidean distance between p and q.

The problem of constructing spanners has been investigated by many researchers. Keil and Gutwin [8] showed that for any $t > 1$, and any set S of n points in the plane, a t-spanner for S having $O(n)$ edges can be constructed in $O(n \log n)$ time. Salowe [9], Vaidya [10] and Callahan and Kosaraju [3] showed the same result for any fixed dimension d. Later work concentrated on constructing spanners that have other properties, such as low degree and low total edge length. For example, Das and Narasimhan [7] gave an $O(n \log^2 n)$-time algorithm that constructs for any set S of n points in $\mathbb{R}^d$, and any constant $t > 1$, a t-spanner for S in which the degree of every point is bounded by a constant, and whose total edge length is proportional to the weight of a minimum spanning tree of S. The time complexity was later improved by Arya et al. [1] to $O(n \log n)$.

In this paper, we show that it is possible to incorporate *fault-tolerance* into such geometric networks while still maintaining many other desirable properties like small weight and degree. Fault tolerance is intimately related to the graph-theoretic concept of *connectivity.* The edge (vertex) connectivity of a graph is the minimum number of edges (vertices) that need to be removed in order to disconnect it. Fault-tolerant networks are usually designed by making them highly connected.

We construct geometric networks that are more than just resilient to edge or vertex faults. Our networks have the property that after removing at most k edges (or vertices), the remaining graph still contains "short" paths between each pair of points. Before we can define this formally, we have to introduce the following notation.

If S is a set of points, then K_S denotes the complete Euclidean graph on S. Let $G = (S, E)$ be a graph, and let E' be a subset of E. Then $G \setminus E'$ denotes the graph $(S, E \setminus E')$. Similarly, if S' is a subset of S, then $G \setminus S'$ is the graph with vertex set $S \setminus S'$, and edge set the set of all edges of E that have both endpoints in $S \setminus S'$.

Definition 1 *Let S be a set of n points in $\mathbb{R}^d$, $t > 1$ a real number, and k an integer such that $1 \leq k \leq n-2$. A graph $G = (S, E)$ is called a k-**edge fault-tolerant** t-**spanner** for S, or (k,t)-**EFTS**, if the following holds for each subset E' of E having size at most k:*

- *For each pair p and q of distinct points in S, the graph $G \setminus E'$ contains a pq-path having length at most t times the length of a shortest pq-path in the graph $K_S \setminus E'$.*

*Similarly, G is called a k-**vertex fault-tolerant** t-**spanner** for S, or (k,t)-**VFTS**, if the following holds for each subset S' of S having size at most k:*

- *The graph $G \setminus S'$ is a t-spanner for the points of $S \setminus S'$.*

Note that in a fault-tolerant t-spanner, our definition insists that between every pair of points there is a path whose length is at most t times the best possible path under the circumstances, i.e., the shortest path in the graph $K_S \setminus E'$ or $K_S \setminus S'$. To our knowledge, fault-tolerant spanners have not been investigated before.

1.1 Our results

We will show in Section 4 that a (k,t)-EFTS having $O(k^2 n)$ edges can be constructed in $O(n \log^{d-1} n + kn \log\log n + k^2 n)$ time, where the constant factors only depend on d and t. Our construction is based on the *well-separated pair decomposition* (WSPD) of Callahan and Kosaraju [4]. They show in [3] that a t-spanner with $O(n)$ edges can be obtained from this decomposition. We extend this result to fault-tolerant spanners. Our construction is simple, but the correctness proof is quite intricate.

The construction needs the notion of a "neighbor" of a pair of points. Given two distinct points a and b in $\mathbb{R}^d$, a neighbor of (a,b) is a point $c \in S \setminus \{a,b\}$ for which $|ac| + |cb|$ is minimal. This notion generalizes to k neighbors in a straightforward way. The problem is that it is not known how to compute such neighbors in polylogarithmic time while using close to linear space. For our purposes, however, it suffices to find an approximation to these neighbors. For this reason, we develop the notion of *strong* approximate neighbors of a pair of points, which is interesting in its own right. In Section 3, we show how to compute k strong approximate neighbors for an off-line batch of m pairs (a,b), in $O((n+m)(\log(n+m) + \log^{d-1}) + km(\log\log n + \log k))$ total time. Our algorithm is based on range trees, and is similar to that of computing the Θ-graph (see [6, 8, 2]), which is in fact a t-spanner. Again, the algorithm is simple, but the correctness proof is non-trivial.

In Section 5, we consider tolerance to vertex faults and show that a (k,t)-VFTS having $O(k^2 n)$ edges can be constructed in $O(n \log n + k^2 n)$ time. Again, the constant factors only depend on d and t. This construction is also based on well-separated pairs, but is much simpler than that for edge fault-tolerant spanners.

In Section 6, we consider fault-tolerant spanners having low total edge length. The main result is a simple construction that transforms any t-spanner G_0 into a (k,t)-EFTS G. In fact, the resulting graph G is also a (k,t)-VFTS. If the degree of each vertex in G_0 is bounded by D, then each vertex of G has degree $O(D^{k+1})$. Moreover, the total edge length of G is proportional to $k \cdot D^k$ times that of G_0. The running time of the algorithm that transforms G_0 into G is bounded by $O(D^{k+1} n)$. Combining a result from Arya et al. [1] with our transformation of Section 6 gives an algorithm that constructs in $O(n \log n + D^{k+1} n)$ time, a (k,t)-EFTS (or (k,t)-VFTS) whose total edge length is proportional to $k \cdot D^k$ times the weight of a minimum spanning tree of S, and in which each point has degree $O(D^{k+1})$. If k is a constant, then this result is optimal, by a result of Chen et al. [5].

2 Preliminaries

Since our construction uses WSPD, we recall two definitions from [4].

Definition 2 *Let $s > 0$ be a real number, called the* separation constant, *and let A and B be two finite sets of points in $\mathbb{R}^d$. We say that A and B are* well-separated w.r.t. s, *if there are two disjoint d-dimensional balls C_A and C_B, having the same radius, such that (i) C_A contains A, (ii) C_B contains B, and (iii) the distance between C_A and C_B is at least equal to s times the radius of C_A.*

Definition 3 *[4] Let S be a set of n points in $\mathbb{R}^d$, and $s > 0$ a real number. A* well-separated pair decomposition *(WSPD) for S (w.r.t. s) is a sequence of pairs of non-empty subsets of S, $\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\}$, such that*

1. *for each unordered pair $\{p, q\}$ of distinct points of S, there exists exactly one pair $\{A_i, B_i\}$ in the sequence, such that $p \in A_i$ and $q \in B_i$, and*
2. *A_i and B_i are well-separated w.r.t. s, for all $i = 1, 2, \ldots, m$.*

The integer m is called the *size* of the WSPD. In [4], it is shown that in $O(n \log n + \alpha_{ds} n)$ time, a WSPD for S having size at most $\alpha_{ds} n$ can be computed. For a large separation constant s, the value of α_{ds} is proportional to $((c+1)s)^d$ for some constant c. The following lemma shows that all edges connecting points from the two sets of a well-separated pair are approximately of the same length. The proof is omitted here.

Lemma 1 *Let $s > 0$ be a real number, and let A and B be two non-empty sets of points in $\mathbb{R}^d$ that are well-separated w.r.t. s. Let a and a' be two points in A, b and b' two points in B, and v an arbitrary point in $\mathbb{R}^d$. Then $|av| + |vb| \leq (1 + 4/s)(|a'v| + |vb'|)$.*

Let S be a set of n points in $\mathbb{R}^d$, and let a and b be two distinct points in $\mathbb{R}^d$. A point $c \in S \setminus \{a, b\}$ is called a *neighbor of the pair* (a, b), if $|ac| + |cb| = \min\{|ax| + |xb| : x \in S \setminus \{a, b\}\}$. More generally, let k be an integer such that $1 \leq k \leq n - 2$. A sequence $c_1, c_2, \ldots, c_k$ of pairwise distinct points of $S \setminus \{a, b\}$ is called a sequence of k *neighbors of the pair* (a, b), if the values $|ac_i| + |c_i b|$, $1 \leq i \leq k$, are the k smallest elements in the set $\{|ax| + |xb| : x \in S \setminus \{a, b\}\}$. We will always assume that the c_i's are numbered in such a way that $|ac_i| + |c_i b| \leq |ac_{i+1}| + |c_{i+1} b|$ for all $1 \leq i < k$. We extend the notion of "neighbors" to that of "strong neighbors". In the definition below, condition 2 reflects the fact that we only have "approximate" neighbors, while condition 3 is the reason for the use of the word "strong".

Definition 4 *Let $c_1, c_2, \ldots, c_k$ be a sequence of k neighbors of the pair (a, b), and $t > 1$ a real number. A sequence $c'_1, c'_2, \ldots, c'_k$ of pairwise distinct points of $S \setminus \{a, b\}$ is called a sequence of* strong t-approximate neighbors of the pair (a, b), *if*

1. *$|ac'_i| + |c'_i b| \leq |ac'_{i+1}| + |c'_{i+1} b|$ for all $1 \leq i < k$,*
2. *$|ac_i| + |c_i b| \leq |ac'_i| + |c'_i b| \leq t\,(|ac_i| + |c_i b|)$ for all $1 \leq i \leq k$, and*
3. *for all $1 \leq i < j \leq k$, if $|ac'_j| + |c'_j b| > t\,(|ac'_i| + |c'_i b|)$, then $c_i \in \{c'_1, c'_2, \ldots, c'_k\}$.*

3 Computing strong approximate neighbors

As mentioned earlier, the notion of strong approximate neighbors of a pair of points was introduced because they can be computed more efficiently than exact neighbors of a pair of points. The discussion below followed by lemma 3 suggests an algorithm to compute strong approximate neighbors of a pair of points.

Our construction of strong approximate neighbors uses the notion of cones. A *(simplicial) cone* is the intersection of d half-spaces in $\mathbb{R}^d$. The hyperplanes that bound these half-spaces are assumed to be in general position, in the sense that their intersection is a point, called the *apex* of the cone.

Let θ be a fixed angle such that $0 < \theta \leq \pi$. Let $\mathcal{C}$ be a collection of cones such that (i) each cone has its apex at the origin, (ii) each cone has *angular diameter* at most θ, i.e., for any two points p and q inside a cone, the vectors $\vec{p}$ and $\vec{q}$ make an angle between zero and θ, and (iii) all cones cover $\mathbb{R}^d$. In Yao [11], it is shown how such a collection $\mathcal{C}$, consisting of $(c/\theta)^{d-1}$ cones, for some constant c, can be constructed.

For each cone $C \in \mathcal{C}$, let l_C be a fixed ray that emanates from the origin and that is contained in C. Let a be any point in $\mathbb{R}^d$. We define $C_a := C + a := \{x + a : x \in C\}$, i.e., C_a is the cone obtained by translating C such that its apex is at a. Similarly, we define $l_{C,a} := l_C + a$. Hence, $l_{C,a}$ is a ray that emanates from a and that is contained in the translated cone C_a.

Lemma 2 ([2]) *Assume that $0 < \theta < \pi/4$. Let a and b be two distinct points in $\mathbb{R}^d$, and let C be the cone of $\mathcal{C}$ such that $b \in C_a$. Let c be any point in C_a such that the projection of c onto the ray $l_{C,a}$ is at least as close to a as the projection of b onto $l_{C,a}$. Then*

$$|cb| \leq |ab| - (\cos\theta - \sin\theta)|ac|.$$

Let S be a set of n points in $\mathbb{R}^d$, a and b two distinct points in $\mathbb{R}^d$, $t > 1$ a real constant, and k an integer such that $1 \leq k \leq n-2$. Let θ be an angle such that $0 < \theta < \pi/4$ and $t \geq 1/(\cos\theta - \sin\theta)$.

For each cone C of $\mathcal{C}$, let $S_{C,a}$ be the set of all points of $S \setminus \{a, b\}$ that are contained in the translated cone C_a. Let $Y_{C,a}$ be the set consisting of the $\min(k, |S_{C,a}|)$ points of $S_{C,a}$ whose orthogonal projections onto $l_{C,a}$ are closest to a.

Let $Z_a := \cup_C Y_{C,a}$. Finally, let $c'_1, c'_2, \ldots, c'_k$ be k pairwise distinct points of Z_a such that

1. $|ac'_i| + |c'_i b| \leq |ac'_{i+1}| + |c'_{i+1} b|$ for all $1 \leq i < k$, and
2. the values $|ac'_i| + |c'_i b|$, $1 \leq i \leq k$, are the k smallest elements in the set $\{|ac| + |cb| : c \in Z_a\}$.

Lemma 3 *$c'_1, c'_2, \ldots, c'_k$ are k strong t-approximate neighbors of (a, b).*

Proof. Let $c_1, c_2, \ldots, c_k$ be the sequence of k exact neighbors of the pair (a, b). Recall that these points are numbered such that $|ac_i| + |c_i b| \leq |ac_{i+1}| + |c_{i+1} b|$ for all $1 \leq i < k$. We will prove that the points $c'_1, c'_2, \ldots, c'_k$ satisfy the three conditions in Definition 4. The first condition obviously holds. Since the points c_i, $1 \leq i \leq k$, are the exact neighbors of (a, b), and the points c'_i, $1 \leq i \leq k$, belong to $S \setminus \{a, b\}$, it is clear that $|ac_i| + |c_i b| \leq |ac'_i| + |c'_i b|$ for all $1 \leq i \leq k$. To prove the second inequality in the second condition of Definition 4, we define a sequence $c''_1, c''_2, \ldots, c''_k$ of points, as follows.

Let C be any cone of $\mathcal{C}$, and let m be the number of points of the set $\{c_1, c_2, \ldots, c_k\}$ that are contained in the translated cone C_a. For any point $c \in C_a$, let $d'(a, c)$ denote the Euclidean distance between a and the projection of c onto the ray $l_{C,a}$. Let $i_1, i_2, \ldots, i_m$ be the indices such that $c_{i_1}, c_{i_2}, \ldots, c_{i_m} \in C_a$, and $d'(a, c_{i_1}) \leq d'(a, c_{i_2}) \leq \ldots \leq d'(a, c_{i_m})$. Note that the indices i_j are, in general, not sorted. For $1 \leq j \leq m$, let c''_{i_j} be the point in $S_{C,a}$ such that $d'(a, c''_{i_j})$ is the j-th smallest element in the set $\{d'(a, c) : c \in S_{C,a}\}$.

Repeating this for all cones C of $\mathcal{C}$ gives us k pairwise distinct points $c''_1, c''_2, \ldots, c''_k$ which are all contained in the set Z_a. Our first claim is that

$$d'(a, c''_i) \leq d'(a, c_i), \tag{1}$$

for all $1 \leq i \leq k$. To prove this, let C be the cone such that $c_i \in C_a$. Let j be the index such that $d'(a, c_i)$ is the j-th smallest element in the set $\{d'(a, c) : c \in \{c_1, c_2, \ldots, c_k\} \cap C_a\}$. We have chosen c''_i in such a way that $d'(a, c''_i)$ is the j-th smallest element in the set $\{d'(a, c) : c \in S_{C,a}\}$. This implies that (1) holds.

Let $1 \leq j \leq k$. We claim that

$$|ac''_j| + |c''_j b| \leq t\,(|ac_j| + |c_j b|). \tag{2}$$

To prove this, we note that Lemma 2 and (1) imply that $|c''_j c_j| \leq |ac_j| - (\cos\theta - \sin\theta)|ac''_j|$. Therefore,

$$\begin{aligned} & |ac''_j| + |c''_j b| \\ & \quad \leq \quad |ac''_j| + |c''_j c_j| + |c_j b| \\ & \quad \leq \quad \frac{|ac_j| - |c''_j c_j|}{\cos\theta - \sin\theta} + |c''_j c_j| + |c_j b|, \end{aligned}$$

from which (2) follows. Now let $1 \leq j \leq i$. Then $|ac''_j| + |c''_j b| \leq t(|ac_j| + |c_j b|) \leq t(|ac_i| + |c_i b|)$. Hence, the set Z_a contains at least i points c such that $|ac| + |cb| \leq t(|ac_i| + |c_i b|)$. Therefore, we must have $|ac'_i| + |c'_i b| \leq t(|ac_i| + |c_i b|)$. Hence, we have shown that the second condition in Definition 4 is satisfied.

It remains to prove the third condition. Let $1 \leq i < j \leq k$ be indices, and assume that $|ac'_j| + |c'_j b| > t(|ac'_i| + |c'_i b|)$. We will show that $c_i \in \{c'_1, c'_2, \ldots, c'_k\}$.

Let C be the cone such that $c_i \in C_a$. We claim that $d'(a, c_i)$ belongs to the $\min(k, |S_{C,a}|)$ smallest elements in the set $\{d'(a,c) : c \in S_{C,a}\}$. If this is true, then c_i is clearly contained in Z_a. Hence, since c_i is the i-th exact neighbor of (p,q), this point must be an element of the set $\{c'_1, c'_2, \ldots, c'_k\}$.

We prove the claim by contradiction. If the claim is not true, then the set $S_{C,a}$ contains more than k elements, and $d'(a, c_i)$ has rank larger than k in the set $\{d'(a,c) : c \in S_{C,a}\}$. Let c be any point in $S_{C,a}$ such that $d'(a,c)$ has rank at most k in the set $\{d'(a,c) : c \in S_{C,a}\}$. Then, c is contained in the set Z_a, and $d'(a,c) \leq d'(a, c_i)$. The latter inequality implies that $|ac| + |cb| \leq t(|ac_i| + |c_i b|)$. (The proof is the same as that of (2).) Hence, $|ac| + |cb| \leq t(|ac_i| + |c_i b|) \leq t(|ac'_i| + |c'_i b|) < |ac'_j| + |c'_j b| \leq |ac'_k| + |c'_k b|$. This shows that the set Z_a contains at least k points c such that $|ac| + |cb| < |ac'_k| + |c'_k b|$, which is clearly a contradiction. ∎

Later, we need an algorithm that solves the following problem: Given a set S of n points in $\mathbb{R}^d$, and an *off-line* sequence F of m pairs of distinct points in $\mathbb{R}^d$, compute for each pair (a,b) in F, the sequence $c'_1, c'_2, \ldots, c'_k$ of strong t-approximate neighbors of Lemma 3.

If $k = 1$, then this problem is basically the same as that of computing the so-called Θ-graph. In [6, 8, 2], efficient algorithms are given that use range trees for constructing this graph. Our algorithm is a direct generalization of the algorithm in [2], and, therefore, we omit the details from this abstract.

Theorem 1 *Let S be a set of n points in $\mathbb{R}^d$, F an off-line sequence of m pairs of distinct points in $\mathbb{R}^d$, $t > 1$ a real constant, and k an integer such that $1 \leq k \leq n-2$. In*

$$O(\beta_{dt}((n+m)(\log(n+m) + \log^{d-1} n) + km(\log\log n + \log k)))$$

time, we can compute for each pair (a,b) in F the sequence of k strong t-approximate neighbors of Lemma 3. If $t \downarrow 1$, then $\beta_{dt} \sim (c/(t-1))^{d-1}$ for some constant c.

4 An edge fault-tolerant spanner

In this section, we first describe how to construct a (k,t)-EFTS for a given set of points. Let S be a set of n points in $\mathbb{R}^d$, $t > 1$ a real constant, and k an integer such that $1 \leq k \leq n-2$. Consider an arbitrary WSPD $\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\}$ for S, with separation constant $s = \max\left(\frac{8t+4}{t-1}, \frac{4}{t^{1/12}-1}\right)$. Our graph G has the points of S as its vertices. The edge set of G is $E := \cup_i E_i, 1 \leq i \leq m$, where E_i is a set of edges joining points in A_i and points in B_i, as defined below. Let $1 \leq i \leq m$, and consider the well-separated pair $\{A_i, B_i\}$. We assume without loss of generality that $|A_i| \geq |B_i|$. To define E_i, we distinguish four cases.

Case 1: $|B_i| \geq k+1$. Choose $k+1$ arbitrary, but pairwise distinct points $a_j \in A_i$, $1 \leq j \leq k+1$, and $k+1$ arbitrary, but pairwise distinct points $b_j \in B_i$, $1 \leq j \leq k+1$. The edge set E_i consists of the $k+1$ edges (a_j, b_j), $1 \leq j \leq k+1$.

Case 2: $|B_i| \leq k$ and $|A_i| \geq k+1$. Choose $k+1$ arbitrary, but pairwise distinct points $a_j \in A_i$, $1 \leq j \leq k+1$. Let $B_i = \{b_1, b_2, \ldots, b_x\}$, where $x = |B_i| \leq k$. The edge set E_i consists of the $x(k+1)$ edges (a_j, b_l), $1 \leq j \leq k+1$, $1 \leq l \leq x$.

Case 3: $|A_i| \leq k$ and $|A_i| + |B_i| \geq k+2$. In this case, the set E_i is defined as the edge set of the complete Euclidean graph on the points of $A_i \cup B_i$.

Case 4: $|A_i| + |B_i| \leq k+1$. Choose an arbitrary point $a \in A_i$, and an arbitrary point $b \in B_i$. Let $c'_1, c'_2, \ldots, c'_k$ be a sequence of k strong $t^{1/3}$-approximate neighbors of the pair (a,b) in the set S. The set E_i is defined as the edge set of the complete Euclidean graph on the points of $A_i \cup B_i \cup \{c'_1, c'_2, \ldots, c'_k\}$.

4.1 The graph G is a (k,t)-EFTS for S

We now prove that the above construction does have the requisite properties. The following obvious but useful lemma is stated without proof.

Lemma 4 *Let $n \geq 3$, K_n the complete graph on n vertices, and E an arbitrary set of at most $n-2$ edges from K_n. For any two vertices p and q, there is a pq-path in the graph $K_n \setminus E$ having at most two edges.*

Let p and q be two distinct points of S, and let E' be an arbitrary subset of E of size at most k. We will prove that the graph $G \setminus E'$ contains a pq-path having length at most t times the length of a shortest pq-path in the graph $K_S \setminus E'$. Let i, $1 \leq i \leq m$, be the index such that $p \in A_i$ and $q \in B_i$.

Lemma 5 *Assume that $|A_i| + |B_i| \leq k+1$. Then there is a pq-path in $G \setminus E'$ of length at most t times the length of a shortest pq-path in $K_S \setminus E'$.*

Proof. First note that we are in Case 4 of the construction of G. Consider the points $a \in A_i$ and $b \in B_i$, and the sequence $c'_1, c'_2, \ldots, c'_k$ of k strong $t^{1/3}$-approximate neighbors of the pair (a, b) that were chosen for the construction of the edge set E_i. Recall that we number these points such that $|ac'_j| + |c'_j b| \leq |ac'_{j+1}| + |c'_{j+1} b|$ for all $1 \leq j < k$. By construction, G contains the complete Euclidean graph on the points of $A_i \cup B_i \cup \{c'_1, c'_2, \ldots, c'_k\}$. Let $c_1, c_2, \ldots, c_k$ be the sequence of k exact neighbors of the pair (a, b). These points are numbered such that $|ac_j| + |c_j b| \leq |ac_{j+1}| + |c_{j+1} b|$ for all $1 \leq j < k$.

Let P be a shortest pq-path in the graph $K_S \setminus E'$. We will prove that $G \setminus E'$ contains a pq-path of length at most t times the length of P. First, we claim that there is a pq-path P' in $K_S \setminus E'$, whose vertices are contained in $V := \{p, q, a, b, c_1, c_2, \ldots, c_k\}$, and that has length at most $(1 + 4/s)^2$ times the length of P. Clearly, this is true if all vertices of P are contained in V. So, assume that P contains a vertex v not belonging to V. Then, by the triangle inequality, P has length at least $|pv| + |vq|$, which, by Lemma 1, is at least equal to $\frac{1}{1+4/s}(|av| + |vb|)$. Since $v \notin \{a, b, c_1, c_2, \ldots, c_k\}$, we have $|av| + |vb| \geq |ac_k| + |c_k b|$. Hence, the length of P satisfies

$$|P| \geq \frac{1}{1+4/s}(|ac_k| + |c_k b|).$$

By Lemma 4, the graph $K_S \setminus E'$ contains a pq-path P' that consists of (i) the single edge (p, q), (ii) two edges (p, a) and (a, q), (iii) two edges (p, b) and (b, q), or (iv) two edges (p, c_j) and (c_j, q) for some index $1 \leq j \leq k$.

If (i) holds, then P' has length $|pq|$, which is clearly at most $(1 + 4/s)^2 |P|$. Assume that (ii) holds. Since the sets A_i and B_i are well-separated, there are two disjoint balls C_{A_i} and C_{B_i} having the same radius, say ρ, that contain A_i and B_i, respectively, and that have distance at least $s\rho$. In particular, we have $|ap| \leq 2\rho$ and $|pq| \geq s\rho$. This, together with the triangle inequality, implies that

$$\begin{aligned} |P'| &\leq |pa| + |ap| + |pq| \\ &\leq 4\rho + |pq| \\ &\leq (1 + 4/s)\,|pq| \\ &\leq (1 + 4/s)^2\,|P|. \end{aligned}$$

If (iii) holds, then it follows in exactly the same way that $|P'| \leq (1 + 4/s)^2 |P|$.

Assume that (iv) holds. Then P' has length $|pc_j| + |c_j q|$, which, by Lemma 1, is at most equal to $(1 + 4/s)\,(|ac_j| + |c_j b|)$. Hence, since $j \leq k$,

$$|P'| \leq (1 + 4/s)\,(|ac_k| + |c_k b|) \leq (1 + 4/s)^2\,|P|.$$

This proves the claim.

In the rest of the proof, we will show that $G \setminus E'$ contains a pq-path Q of length at most t times the length of P. If all vertices of P' are contained in the set $\{p, q, a, b, c'_1, c'_2, \ldots, c'_k\}$, then P' is a pq-path in the graph $G \setminus E'$, and we take $Q = P'$. Our choice of s implies that $(1 + 4/s)^2 \leq t$. Hence, $|Q| \leq t|P|$. Assume that the path P' contains a vertex w that is not contained in $\{p, q, a, b, c'_1, c'_2, \ldots, c'_k\}$. Let l be the index such that $w = c_l$. Applying the triangle inequality, we see that

$$|P'| \geq |pw| + |wq| = |pc_l| + |c_l q|.$$

Note that the set $\{p, q, a, b, c'_1, c'_2, \ldots, c'_k\}$ has size at least $k + 2$, and that G contains the complete graph on this set. Therefore, by Lemma 4, the graph $G \setminus E'$ contains a pq-path Q consisting of (i) the edge (p, q), (ii) two edges (p, a) and (a, q), (iii) two edges (p, b) and (b, q), or (iv) two edges (p, c'_j) and (c'_j, q), for some index j.

If (i) holds, then Q has length at most t times the length of P. If (ii) holds, then Q has length $|pa| + |aq|$. As in (ii) above, it follows that $|Q| \leq (1+4/s)|pq| \leq (1+4/s)|P|$. Our choice of s implies that that $1+4/s \leq t$. That is, Q has length at most $t|P|$. If (iii) holds, then it follows in the same way that the path Q has length at most $t|P|$. Finally, assume that (iv) holds. If $j \leq l$, then

$$|Q| = |pc'_j| + |c'_j q|$$

$$
\begin{aligned}
&\leq (1+4/s)\,(|ac_j'|+|c_j'b|)\\
&\leq (1+4/s)\,(|ac_l'|+|c_l'b|)\\
&\leq t^{1/3}(1+4/s)\,(|ac_l|+|c_lb|)\\
&\leq t^{1/3}(1+4/s)^2\,(|pc_l|+|c_lq|)\\
&\leq t^{1/3}(1+4/s)^2\,|P'|\\
&\leq t^{2/3}(1+4/s)^2\,|P'|\\
&\leq t^{2/3}(1+4/s)^4\,|P|.
\end{aligned}
$$

Note that we have chosen s such that $s \geq 4/(t^{1/12}-1)$, which is equivalent to $(1+4/s)^4 \leq t^{1/3}$. Therefore, the length of Q is at most t times the length of P. It remains to consider the case when $l < j$. We claim that

$$|ac_j'|+|c_j'b| \leq t^{1/3}\,(|ac_l'|+|c_l'b|).$$

Indeed, if this is not true, then the third condition in Definition 4 implies that $w = c_l \in \{c_1', c_2', \ldots, c_k'\}$, which is a contradiction. Hence,

$$
\begin{aligned}
|Q| &= |pc_j'|+|c_j'q|\\
&\leq (1+4/s)\,(|ac_j'|+|c_j'b|)\\
&\leq t^{1/3}(1+4/s)\,(|ac_l'|+|c_l'b|)\\
&\leq t^{2/3}(1+4/s)\,(|ac_l|+|c_lb|)\\
&\leq t^{2/3}(1+4/s)^2\,(|pc_l|+|c_lq|)\\
&\leq t^{2/3}(1+4/s)^2\,|P'|\\
&\leq t^{2/3}(1+4/s)^4\,|P|.
\end{aligned}
$$

Then in the same way as above, we obtain $|Q| \leq t|P|$. ■

Now we can give the complete proof of the fact that the graph $G \setminus E'$ contains a pq-path of length at most t times the length of a shortest pq-path in $K_S \setminus E'$. The proof is by induction on the rank of the distance $|pq|$ in the sorted sequence of $\binom{n}{2}$ distances in S. If p,q is a closest pair in S, then $A_i = \{p\}$ and $B_i = \{q\}$, and the claim follows from Lemma 5. Assume that p,q is not a closest pair. Moreover, assume that for any pair $a,b \in S$ with $|ab| < |pq|$, the graph $G \setminus E'$ contains an ab-path of length at most t times the length of a shortest ab-path in $K_S \setminus E'$. Since the sets A_i and B_i are well-separated, there are two balls C_{A_i} and C_{B_i} having the same radius, say ρ, that contain A_i and B_i, respectively, and that have distance at least $s\rho$. We have to consider Cases 1–4. Case 4 has been handled already. In this abstract, we only treat Case 1. So assume that $|A_i| \geq |B_i| \geq k+1$. Consider the $k+1$ points $a_j \in A_i$, $1 \leq j \leq k+1$, and the $k+1$ points $b_j \in B_i$, $1 \leq j \leq k+1$, that were chosen in the construction of G.

Lemma 6 *There is an index j, $1 \leq j \leq k+1$, such that the graph $G \setminus E'$ contains (i) the edge (a_j, b_j), (ii) a path P between p and a_j of length at most $4t\rho$, and (iii) a path Q between q and b_j of length at most $4t\rho$.*

Proof. We will first prove that for each j such that (a_j, b_j) is an edge in $G \setminus E'$, there is a pa_j-path P_j in $G \setminus E'$ of length at most $4t\rho$.

First assume that $p \notin \{a_1, a_2, \ldots, a_{k+1}\}$. Since E' has size at most k, there is an index j, such that (a_j, b_j) is an edge in $G \setminus E'$. Let j be an arbitrary index having this property.

Since $|pq| \geq s\rho$, $|pa_j| \leq 2\rho$, and $s > 2$, we must have $|pa_j| < |pq|$. Therefore, by the induction hypothesis, the graph $G \setminus E'$ contains a path P_j between p and a_j having length at most t times the length of a shortest pa_j-path in $K_S \setminus E'$. Since A_i contains at least $k+2$ points, Lemma 4 implies that the graph $K_{A_i} \setminus E'$ contains a pa_j-path P' having at most two edges. Each edge on this path is contained in the graph $K_S \setminus E'$, and has length at most 2ρ. Hence, P' has length at most 4ρ. It follows that the shortest pa_j-path in $K_S \setminus E'$ has length at most 4ρ. Therefore, $|P_j| \leq 4t\rho$.

Next assume that $p \in \{a_1, a_2, \ldots, a_{k+1}\}$. Let l be the index such that $p = a_l$. If $G \setminus E'$ contains the edge (a_l, b_l), then we take for P_l the empty pa_l-path, having length zero. So assume that (a_l, b_l) is not an edge of $G \setminus E'$. Let j be an arbitrary index such that $G \setminus E'$ contains the edge (a_j, b_j). The induction hypothesis implies that $G \setminus E'$ contains a path P_j between p and a_j having length at most t times the length of a shortest pa_j-path in $K_S \setminus E'$.

Since E' contains (a_l, b_l), at most $k-1$ edges of the subgraph K_{A_i} are contained in E'. Then, since A_i contains at least $k+1$ points, Lemma 4 implies that the graph $K_{A_i} \setminus E'$ contains a pa_j-path having at most two edges. Each edge on this path is contained in $K_S \setminus E'$, and has length at most 2ρ. Therefore, path P_j has length at most $4t\rho$.

This completes the proof of the claim mentioned in the beginning of this proof. Now assume that $q \notin \{b_1, b_2, \ldots, b_{k+1}\}$. Let j be an arbitrary index such that (a_j, b_j) is an edge in $G \setminus E'$. Then, in exactly the same way as above, it can be shown that $G \setminus E'$ contains a qb_j-path Q_j, having length at most $4t\rho$. Hence, the index j, and the paths P_j and Q_j satisfy the three conditions mentioned in the lemma. The case when $q \in \{b_1, b_2, \ldots, b_{k+1}\}$ can be handled similarly. ∎

We can now complete the proof for Case 1. Consider the index j, and the corresponding paths P and Q, of Lemma 6. Let R be the pq-path in $G \setminus E'$ obtained by concatenating path P, edge (a_j, b_j), and path Q. Then $|R| \leq t|pq|$. Hence, we have shown that the graph $G = (S, E)$ is a (k,t)-EFTS for S. Note that it has $O(k^2 m)$ edges.

Theorem 2 *Let S be a set of n points in $\mathbb{R}^d$, k an integer such that $1 \leq k \leq n-2$, and $t > 1$ a real constant. In $O\left(\delta_{dt}\left(n \log^{d-1} n + kn \log\log n + k^2 n\right)\right)$ time, a (k,t)-EFTS for S, containing at most $\gamma_{dt} k^2 n$ edges, can be computed. If $t \downarrow 1$, then $\gamma_{dt} \sim (c/(t-1))^d$ and $\delta_{dt} \sim (c/(t-1))^{2d-1}$ for some constant c.*

5 A vertex fault-tolerant spanner

The construction in this section is the same as in Section 4, except that Case 4 is much simpler. Consider an arbitrary WSPD $\{A_1, B_1\}, \{A_2, B_2\}, \ldots, \{A_m, B_m\}$ for S, with separation constant $s = 4(t+1)/(t-1)$. As in Section 4, our graph G has the points of S as its vertices. For each i, $1 \leq i \leq m$, we will define a set E_i of edges. Then, $E := \cup_i E_i$ will be the edge set of G. Let $1 \leq i \leq m$, and consider the well-separated pair $\{A_i, B_i\}$. To define E_i, we distinguish the same four cases as in Section 4. In Case 1, 2, or 3, the edge set E_i is defined as in Section 4. In Case 4, E_i is the edge set of the complete Euclidean graph on the points of $A_i \cup B_i$. The graph G is a (k,t)-VFTS for the set S. The proof is similar to that in Section 4.1 and is omitted here.

Theorem 3 *Let S be a set of n points in $\mathbb{R}^d$, k an integer such that $1 \leq k \leq n-2$, and $t > 1$ a real constant. In $O(n \log n + \gamma_{dt} k^2 n)$ time, a (k,t)-VFTS for S containing at most $\gamma_{dt} k^2 n$ edges can be computed. If $t \downarrow 1$, then $\gamma_{dt} \sim (c/(t-1))^d$ for some constant c.*

6 Fault-tolerant spanners of low weight

We now turn to the problem of constructing fault-tolerant spanners whose total edge length is "small". First, we give a simple transformation that turns any spanner G_0 into a fault-tolerant spanner G. If the degree of G_0 is bounded by D, then the degree of G is proportional to D^{k+1}. Moreover, in this case, the transformation increases the total edge length by at most a factor proportional to $k \cdot D^k$.

Let S be a set of n points in $\mathbb{R}^d$, $t > 1$ a real number, and k an integer such that $2 \leq k \leq n-2$. Let G_0 be an arbitrary t-spanner for S. For each vertex $p \in S$, let $N(p)$ be the set of all vertices of $S \setminus \{p\}$ that are connected to p, in G_0, by a path consisting of at most $k+1$ edges. Define $E_p := \{(p,q) : q \in N(p)\}$. The transformed graph G has the points of S as its vertices, and it has edge set $E := \cup_{p \in S} E_p$. Note that G_0 is a subgraph of G.

Lemma 7 *Let E' be a subset of E having size at most k, and let (p,q) be an edge of $K_S \setminus E'$. The graph $G \setminus E'$ contains a pq-path of length at most t times the Euclidean distance between p and q.*

Proof. Since G_0 is a t-spanner for S, there is a pq-path

$$P = (q_0 = p, q_1, q_2, \ldots, q_l = q)$$

in G_0 of length at most $t|pq|$. We will construct a pq-path Q in $G \setminus E'$ that is a subpath of P. Then, the triangle inequality implies that the length of Q is at most that of P. This will prove the lemma.

First assume that $l \leq k+1$. Then, $q \in N(p)$ and, hence, (p,q) is an edge of G. Since (p,q) is an edge of $K_S \setminus E'$, it is not contained in E'. Therefore, (p,q) is an edge of $G \setminus E'$, and we can take for Q the path consisting of this single edge.

Assume that $k+2 \leq l$. The following algorithm constructs the path $Q = (p_0, p_1, \ldots)$ incrementally.
Step 1: Define $p_0 := p$, $i := 0$, and $j := 0$. Go to Step 2.
Step 2: At this moment, $Q = (p_0, p_1, \ldots, p_i)$ is a path in $G \setminus E'$, j is the index such that $p_i = q_j$, and $j + k + 2 \leq l$. (In particular, $p_i \neq q$.)

If there is an index m, $j+1 \leq m \leq j+k+1$, such that (i) $m + k + 2 \leq l$ and (ii) (q_j, q_m) is an edge of $G \setminus E'$, then go to Step 3. Otherwise, go to Step 4.

Step 3: Define $p_{i+1} := q_m$. Moreover, set $i := i+1$, and $j := m$, and go to Step 2.

Step 4: We know that $p_i = q_j$ and $j + k + 2 \leq l$. Moreover, for all m, $j+1 \leq m \leq j+k+1$, such that (q_j, q_m) is an edge of $G \setminus E'$, we have $m+k+1 \geq l$.

We claim that there is an index m, $j+1 \leq m \leq j+k+1$, such that (q_j, q_m) and (q_m, q) are both edges of $G \setminus E'$.

Assume this claim is true. Then we define $p_{i+1} := q_m$ and $p_{i+2} := q$, and the construction of the pq-path Q is complete.

It remains to prove the claim. First note that all $k+1$ edges (q_j, q_m), $j+1 \leq m \leq j+k+1$, are contained in G. Let

$$x := \left| E' \cap \{(q_j, q_m) : j+1 \leq m \leq j+k+1\} \right|.$$

Let M be the set of indices m, $j+1 \leq m \leq j+k+1$, such that (q_j, q_m) is an edge of $G \setminus E'$. Then M has size $k+1-x$. Let m be an arbitrary element of M. Then our assumption implies that $m+k+1 \geq l$. Therefore, $q \in N(q_m)$, and $(q_m, q) \in G$.

Since the set $\{(q_m, q) : m \in M\}$ has size $k+1-x$, and at most $k-x$ edges of this set are contained in E', there exists an index $m \in M$ such that (q_m, q) is an edge of $G \setminus E'$. This proves the claim.

Why does this algorithm terminate? Each time Step 3 is executed, path Q is extended by a new point. Therefore, at some moment, Step 4 must be executed. At that moment, Q reaches q, and the algorithm terminates. ∎

Lemma 8 *The graph G is a (k,t)-EFTS and a (k,t)-VFTS for S.*

Proof. We only consider the edge tolerance of G. Let E' be a subset of E having size at most k, and let p and q be two distinct points of S. Let $P = (p_0 = p, p_1, p_2, \ldots, p_l = q)$ be a shortest pq-path in $K_S \setminus E'$. Then for each i, $0 \leq i < l$, (p_i, p_{i+1}) is an edge of $K_S \setminus E'$. Hence by Lemma 7, the graph $G \setminus E'$ contains a path Q_i between p_i and p_{i+1} having length at most $t|p_i p_{i+1}|$. Let Q be the concatenation of $Q_0, Q_1, \ldots, Q_{l-1}$. Then, Q is a pq-path in $G \setminus E'$, having length

$$\sum_{i=0}^{l-1} |Q_i| \leq \sum_{i=0}^{l-1} t|p_i p_{i+1}| = t|P|.$$

This proves the lemma. ∎

Lemma 9 *Assume that each point of S has degree at most D in G_0. Then (i) each point of S has degree at most $2 \cdot D^{k+1}$ in G, and (ii) the total edge length of G is at most $8(k+1) \cdot D^k$ times that of G_0.*

Theorem 4 *Let S be a set of n points in $\mathbb{R}^d$, k a positive integer, and $t > 1$ a real constant. There exists a (k,t)-EFTS and (k,t)-VFTS G for S in which each point has degree at most α^{dk}, and whose total edge length is at most α^{dk} times the weight of a minimum spanning tree of S. This graph G can be computed in $O(n \log n + \alpha^{dk} n)$ time. If $t \downarrow 1$, then $\alpha \sim c/(t-1)$ for some constant c.*

Proof. Omitted. ∎

7 Concluding remarks

We have presented efficient algorithms for constructing spanners that are resilient to k vertex or edge faults. In particular, Theorems 2 and 3 give constructions that use a polynomial (i.e, $O(k^2 n)$) number of edges. On the other hand, the construction of Theorem 4 uses a number of edges that is exponential in k. In the latter construction, however, upper bounds on the degree and weight can be guaranteed.

If k is a constant, then the best result is that of Theorem 4. It gives a k-fault-tolerant t-spanner, in which the degree of each vertex is bounded by a constant, and whose weight is proportional to the weight of a minimum spanning tree. Moreover, this spanner can be constructed in $O(n \log n)$ time. Chen et al.[5] showed that constructing *any* t-spanner—that is not necessarily resilient to faults—takes $\Omega(n \log n)$ time in the algebraic computation tree model. Therefore, the result of Theorem 4 is optimal for constant values of k.

Some interesting problems remain to be solved. Any graph on n vertices that remains connected after removing at most k edges or vertices, must have $\Omega(kn)$ edges. The reason is that each vertex in such a graph must have degree at least $k+1$. Is it possible to construct a (k,t)-EFTS or (k,t)-VFTS with $O(kn)$ edges, in $O(n \log n + kn)$ time?

Let k be an even integer, and consider a set A of $1 + k/2$ points that are all very close to the origin. Let B be a set of $n-1-k/2$ points that are all very close together, but at distance roughly one from the origin. Let G be any (k,t)-EFTS or (k,t)-VFTS for the set $S := A \cup B$, where t is a constant close to one. Then, every point of A has to be connected to at least $1 + k/2$ points of B. Hence, G contains $\Omega(k^2)$ edges having length roughly equal to one. On the other hand, a minimum spanning tree of S has weight roughly equal to one.

Is it possible to construct, for any set S of n points in $\mathbb{R}^d$, a (k,t)-EFTS or (k,t)-VFTS, such that each vertex has degree $O(k)$, and the weight of this graph is $O(k^2)$ times the weight of a minimum spanning tree of S? Can such a graph be constructed in $O(n \log n + kn)$ time?

Remark 1 After this paper was submitted, we improved the results that were presented here to spanners that are resilient to both vertex and edge faults. To define this formally, we need the following notation. Let $G = (S, E)$ be a graph, E' a subset of E, and S' a subset of S. Then, $G \setminus (S', E')$ denotes the graph with vertex set $S \setminus S'$, and edge set the set of all edges of $E \setminus E'$ that have both endpoints in $S \setminus S'$.

A graph $G = (S, E)$ is called a k-**fault-tolerant** t-**spanner** for S, or (k,t)-**FTS**, if the following holds for each subset S' of S and each subset E' of E such that $|S'| + |E'| \leq k$:

- For each pair p and q of distinct points in $S \setminus S'$, the graph $G \setminus (S', E')$ contains a pq-path having length at most t times the length of a shortest pq-path in the graph $K_S \setminus (S', E')$.

We proved that any (k,t)-VFTS is also a (k,t)-FTS and, hence, in particular a (k,t)-EFTS. This, together with Theorem 3, leads to an algorithm that computes a (k,t)-FTS with $O(k^2 n)$ edges, in $O(n \log n + k^2 n)$ time.

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