k-Selection; Median

- Select the k-th smallest item in the list
- Naïve Solution
  - Sort;
  - pick the k-th smallest item in sorted list.

    $O(n \log n)$ time complexity
- Randomized solution: Average case $O(n)$
- Improved Solution: worst case $O(n)$

```
QuickSort(A, p, r)
    if (p < r) then
        q = Partition(A, p, r)
        QuickSort(A, p, q)
        QuickSort(A, 1+1, r)

Partition(A, p, r)
  x = A[p]
  i = p-1
  j = r+1
  while TRUE do
      repeat
          j- -
      until (A[j] <= x)
       repeat
          i++
      until (A[i] >= x)
      if (i < j) SWAP(A[i], A[j])
      else return j
```

Partition Procedure Revisited

- The Partition code can be rewritten so that it accepts another parameter, namely, the pivot value. Let's call this new variation as PivotPartition.
- This change does not affect its time complexity.
- RandomizedPartition as used in RandomizedSelect picks the pivot uniformly at random from among the elements in the list to be partitioned.

Randomized Selection

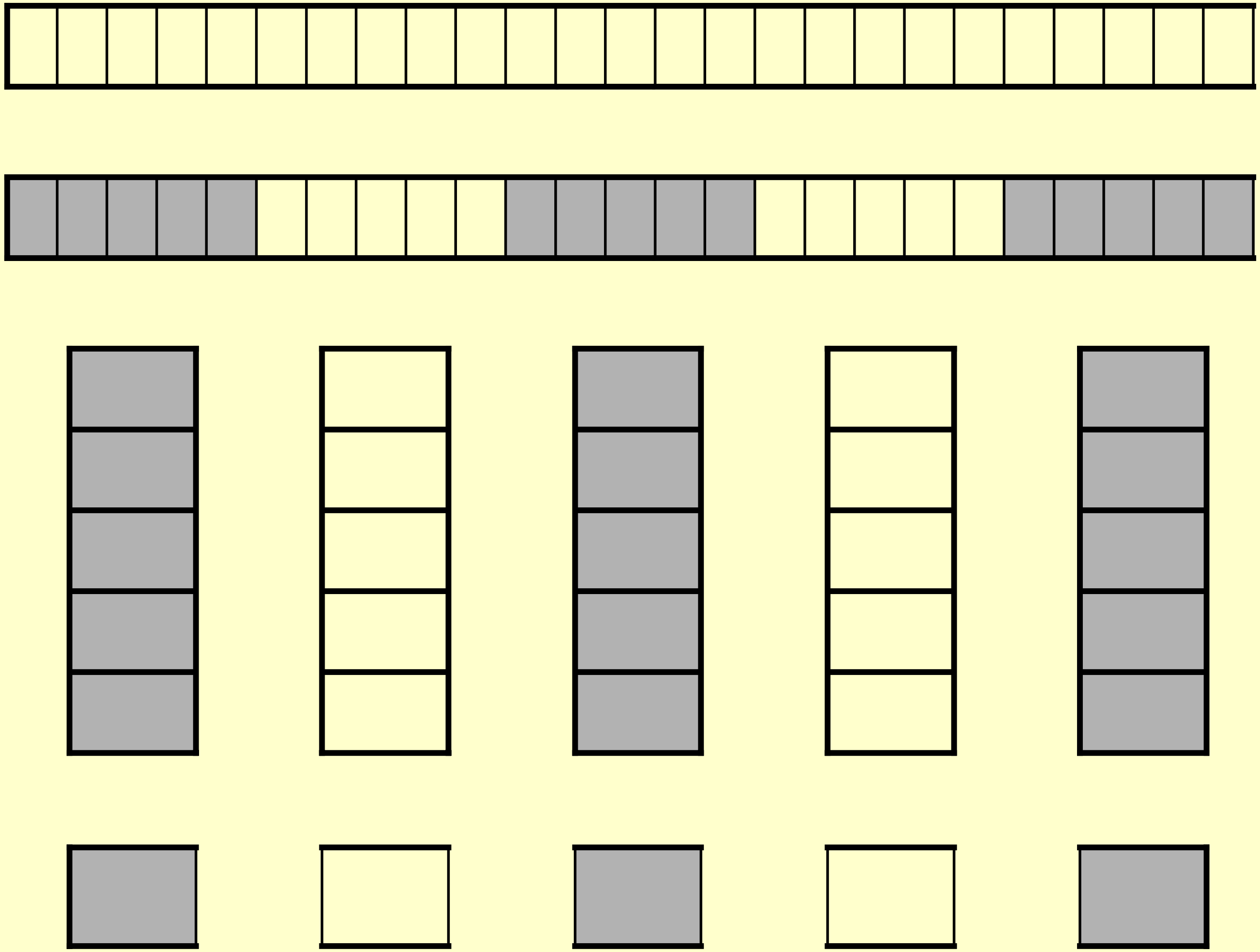
```
RandomizedSelect(A, p, r, i)
    if (p = r) then
        return A[p]
    q = RandomizedPartition(A, p, r)
    k = q - p + 1
    if (i <= k)
        return RandomizedSelect(A, p, q, i)
    else
        return RandomizedSelect(A, q+1, r, i-k)
```

Randomized Selection: Rewritten

```
RandomizedSelect(A, p, r, i)
    if (p = r) then
        return A[p]
    Pivot = A[random(p,r)]
    q = PivotPartition(A, p, r, Pivot)
    k = q – p + 1
    if (i <= k)
        return RandomizedSelect(A, p, q, i)
    else
        return RandomizedSelect(A, q+1, r, i-k)
```

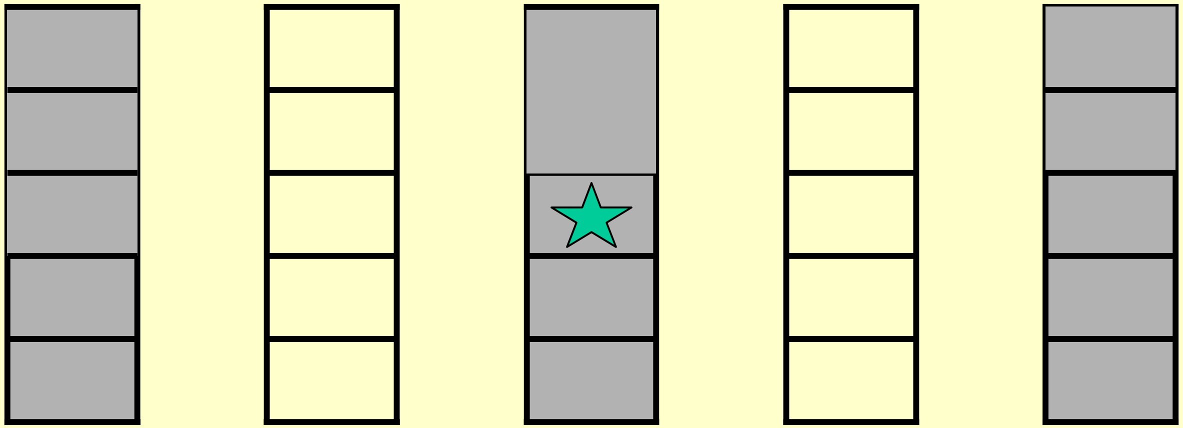
k-Selection & Median: Improved Algorithm

- Start with initial array



k-Selection & Median: Improved Algorithm(Cont'd)

- Use median of medians as pivot



- $T(n) < O(n) + T(n/5) + T(3n/4)$

Improved Selection

```
ImprovedSelect(A, p, r, i)
    if (p = r) then
        return A[p]
    else N = r – p + 1
    Partition A[p..r] into subsets of 5 elements and collect all
     the medians of the subsets in B[1..(N/5)].
    Pivot = ImprovedSelect (B, 1, ⌈N/5⌉, ⌈N/10⌉ )
    q = PivotPartition (A, p, r, Pivot)
    k = q – p + 1
    if (i <= k)
        return ImprovedSelect(A, p, q, i)
    else
        return ImprovedSelect(A, q+1, r, i-k)
```

Upper & Lower Bounds

- Algorithm A solves problem P if it terminates & gives the correct output on every possible input.
- Algorithm A solving problem P has time complexity f(n) if it takes time at most f(n) for every input of length n.
- U(n) is an upper bound on the time complexity of P, if there exists an algorithm A that solves P and has time complexity U(n).
- L(n) is a lower bound on the time complexity of P, if there exists NO algorithm that solves P and has time complexity asymptotically less than L(n).

Upper & Lower Bounds for Maximum

- Naïve Algorithm A solves the Maximum problem, because it terminates in n iterations for every possible input of length n and outputs the correct maximum.
- Naïve Algorithm A has time complexity O(n).
- O(n) is an upper bound on the time complexity of the maximum problem.
- (n-1) is a lower bound on the time complexity of the maximum problem, because there exists NO algorithm that solves it with less than n-1 comparisons.
- WHY? In 1 comparison, at most 1 item is eliminated from being the maximum. How many to eliminate?
- Therefore, no matter how smart you are you cannot design an algorithm that solves the Maximum problem in less than n-1 comparisons on all inputs of length n.

Upper Bound on Sorting n items

- O(n log n) is the upper bound for sorting.
- WHY?
  - HeapSort
  - MergeSort
- What about QuickSort?
  - $O(n^2)$ in the worst case!

Lower Bound for Sorting: Decision Tree Model

- The decision tree model models all comparison-based algorithms that solve the sorting problem. These algorithms perform no other "algebraic" operations on input values. They may perform data movements & other statements.
- Imagine a binary tree that models the algorithm, where
  - each node corresponds to a comparison
  - the edges to the children correspond to the two outcomes of the comparison: YES/NO
  - Leaves correspond to the output. WHAT IS THE OUTPUT?
- Decision tree for InsertionSort on 4 items?
- What can we say about such decision trees?
- Given an input, the algorithm follows a path from the root to a leaf.

Lower Bound for Sorting: Cont'd

- Leaves correspond to outputs.
- Paths correspond to a path followed on a specific input. Time complexity = height of decision tree.
- Different input orders must force different paths or else the output will end up being the same, giving rise to incorrect sorted orders.
- Therefore number of leaves is at least as large as the number of different input orders.
  - HOW MANY?
  - n!
- Height of the decision tree is at least log(n!). Hence lower bound is O(log(n!)) = O(n log n)