Amortized Analysis

- In amortized analysis, we are looking for the time complexity of a sequence of n operations, instead of the cost of a single operation.
- Cost of a sequence of n operations = n $S(n)$, where $S(n)$ = worst case cost of each of the n operations
- ***Amortized Cost*** = $T(n)/n$, where $T(n)$ = worst case total cost of the n operations in the sequence.
- Amortized cost can be small even when some operations in that sequence are expensive. Often, the worst case may not occur in every operation. The cost of expensive operations may be ‘paid for’ by charging to other less expensive operations.

Problem 1: Stack Operations

- Data Structure: Stack
- Operations:
  - *Push(s,x)* : Push object *x* into stack *s*.
    - Cost: T(push)= O(1).
  - *Pop(s) :* Pop the top object in stack *s*.
    - Cost: T(pop)=O(1).
  - *MultiPop(s,k) ;* Pop the top *k* objects in stack *s*.
    - Cost: T(mp) = O(size(s)) worst case
- ***Assumption:*** Start with an empty stack
- ***Simple analysis:*** For N operations, the maximum size of stack is N. Since the cost of *MultiPop* under the worst case is O(N), which is the largest in all three operations, the total cost of N operations must be less than N x T(mp) = $O(N^2)$.

Amortized analysis: Stack Operations

- **Intuition**: Worst case cannot happen all the time!
- **Idea**: pay a dollar for every operation, and then count carefully.
- Suppose we pay 2 dollars for each *Push* operation, one to pay for the operation itself, and another for "future use" (we pin it to the object on the stack).
- When we do *Pop* or *MultiPop* operations to pop objects, instead of paying from our pocket, we pay the operations with the extra dollar pinned to the objects that are being popped.
- So the total cost of N operations must be less than 2 x N
- **Amortized cost** = T(N)/N = 2.

Problem 2: Binary Counter

- Data Structure: binary counter b.
- Operations: Inc(b).
  - Cost of Inc(b) = number of bits flipped in the operation.
- What's the total cost of N operations when this counter counts up to integer N?
- *Approach 1: simple analysis*
  - The size of the counter is log(N). The worst case will be that every bit is flipped in an operation, so for N operations, the total cost under the worst case is O(Nlog(N))

Approach 2: Binary Counter

- **Intuition**: Worst case cannot happen all the time!

000000

000001

000010

000011

000100

000101

000110

000111

Bit 0 flips every time, bit 1 flips every other time, bit 2 flips every fourth time, etc. We can conclude that for bit k, it flips every 2^k time.

So the total bits flipped in N operations, when the counter counts from 1 to N, will be = ?

$$T(N) = \sum_{k=0}^{\log N} \frac{N}{2^k} < N \sum_{k=0}^{\infty} \frac{1}{2^k} = 2N$$

So the amortized cost will be T(N)/N = 2.

Approach 3: Binary Counter

- For k bit counters, the total cost is

  $t(k) = 2 \times t(k-1) + 1$
- So for N operations, $T(N) = t(\log(N))$.
- $t(k) = ?$
- $T(N)$ can be proved to be bounded by $2N$.

Amortized Analysis: Potential Method

- For the n operations, the data structure goes through states: D_0, D_1, D_2, …, D_n with costs c_1, c_2, …, c_n
- Define potential function $\Phi(D_i)$: represents the potential energy of data structure after i_{th} operation.
- The amortized cost of the i_{th} operation is defined by:

$$\hat{c}_i = c_i + \Phi(D_i) - \Phi(D_{i-1})$$

- The total amortized cost is

$$\sum_{i=1}^{n} \hat{c}_i = \sum_{i=1}^{N} \left(c_i + \Phi(D_i) - \Phi(D_{i-1})\right) = \Phi(D_n) - \Phi(D_0) + \sum_{i=1}^{n} c_i$$

$$\sum_{i=1}^{n} c_i = -\left(\Phi(D_n) - \Phi(D_0)\right) + \sum_{i=1}^{n} \hat{c}_i$$

Polynomial-time computations

- An algorithm has time complexity $O(T(n))$ if it runs in time at most $cT(n)$ for every input of length n.
- An algorithm is a polynomial-time algorithm if its time complexity is $O(p(n))$, where $p(n)$ is polynomial in n.

Polynomials

- If f(n) = polynomial function in n,

  then $f(n) = O(n^c)$, for some fixed constant c
- If f(n) = exponential (super-polynomial) function in n,

  then $f(n) = \omega(n^c)$, for any constant c
- Composition of polynomial functions are also polynomial, i.e.,

  f(g(n)) = polynomial if f() and g() are polynomial
- If an algorithm calls another polynomial-time subroutine a polynomial number of times, then the time complexity is polynomial.

The class $\mathcal{P}$

- A problem is in $\mathcal{P}$ if there exists a polynomial-time algorithm that solves the problem.
- Examples of $\mathcal{P}$
  - ***DFS:*** Linear-time algorithm exists
  - ***Sorting:*** O(n log n)-time algorithm exists
  - ***Bubble Sort:*** Quadratic-time algorithm $O(n^2)$
  - ***APSP:*** Cubic-time algorithm $O(n^3)$
- $\mathcal{P}$ is therefore a class of problems (not algorithms)!

The class $\mathcal{NP}$

- A problem is in $\mathcal{NP}$ if there exists a non-deterministic polynomial-time algorithm that solves the problem.
- A problem is in $\mathcal{NP}$ if there exists a (deterministic) polynomial-time algorithm that verifies a solution to the problem.
- All problems in $\mathcal{P}$ are in $\mathcal{NP}$

TSP: Traveling Salesperson Problem

- **Input:**
  - Weighted graph, G
  - Length bound, B
- **Output:**
  - Is there a traveling salesperson tour in G of length at most B?
- Is TSP in $\mathcal{NP}$?
  - YES. Easy to verify a given solution.
- Is TSP in $\mathcal{P}$?
  - OPEN!
  - One of the greatest unsolved problems of this century!
  - Same as asking: **Is $\mathcal{P} = \mathcal{NP}$?**

So, what is *NP-Complete*?

- *NP-Complete* problems are the “hardest” problems in *NP*.
- We need to formalize the notion of “hardest”.

Terminology

- Problem:
  - An abstract problem is a function (relation) from a set I of instances of the problem to a set S of solutions.

    $p: I \rightarrow S$
  - An instance of a problem *p* is obtained by assigning values to the parameters of the abstract problem.
  - Thus, describing the set of all instances (I.e., possible inputs) and the set of corresponding outputs defines a problem.
- Algorithm:
  - An algorithm that solves problem *p* must give correct solutions to all instances of the problem.
- Polynomial-time algorithm:

Terminology (Cont'd)

- Input Length:
  - length of an encoding of an instance of the problem.
  - Time and space complexities are written in terms of it.
- Worst-case time/space complexity of an algorithm
  - Is the maximum time/space required by the algorithm on any input of length n.
- Worst-case time/space complexity of a problem
  - UPPER BOUND: worst-case time complexity of best existing algorithm that solves the problem.
  - LOWER BOUND: (provable) worst-case time complexity of best algorithm (need not exist) that could solve the problem.
  - LOWER BOUND $\leq$ UPPER BOUND
- Complexity Class $\mathcal{P}$:
  - Set of all problems p for which polynomial-time algorithms exist

Terminology (Cont'd)

- Decision Problems:
  - Are problems for which the solution set is {yes, no}
  - Example: Does a given graph have an odd cycle?
  - Example: Does a given weighted graph have a TSP tour of length at most B?
- Complement of a decision problem:
  - Are problems for which the solution is "complemented".
  - Example: Does a given graph NOT have an odd cycle?
  - Example: Is every TSP tour of a given weighted graph of length greater than B?
- Optimization Problems:
  - Are problems where one is maximizing (or minimizing) some objective function.
  - Example: Given a weighted graph, find a MST.
  - Example: Given a weighted graph, find an optimal TSP tour.
- Verification Algorithms:
  - Given a problem instance i and a certificate s, is s a solution for instance i?

Terminology (Cont'd)

- Complexity Class *P* :
  - Set of all problems *p* for which polynomial-time algorithms exist.
- Complexity Class *NP* :
  - Set of all problems *p* for which polynomial-time verification algorithms exist.
- Complexity Class *co-NP* :
  - Set of all problems *p* for which polynomial-time verification algorithms exist for their **complements**, I.e., their complements are in *NP*.

Terminology (Cont'd)

- **Reductions:** $p_1 \rightarrow p_2$
  - A problem p_1 is reducible to p_2, if there exists an algorithm R that takes an instance i_1 of p_1 and outputs an instance i_2 of p_2, with the constraint that the solution for i_1 is YES if and only if the solution for i_2 is YES.
  - Thus, R converts YES (NO) instances of p_1 to YES (NO) instances of p_2.
- Polynomial-time reductions: $p_1 \xrightarrow{P} p_2$
  - Reductions that run in polynomial time.

- If $p_1 \xrightarrow{P} p_2$, then
  - If p_2 is easy, then so is p_1. $p_2 \in \mathcal{P} \Rightarrow p_1 \in \mathcal{P}$
  - If p_1 is hard, then so is p_2. $p_1 \notin \mathcal{P} \Rightarrow p_2 \notin \mathcal{P}$

What are *NP-Complete* problems?

- These are the hardest problems in *NP*.
- A problem p is *NP-Complete* if
  - there is a polynomial-time reduction from **every** problem in *NP* to p.
  - $p \in NP$
- How to prove that a problem is *NP-Complete*?

- **Cook's Theorem**: [1972]
  - The SAT problem is *NP-Complete*.

Steve Cook, Richard Karp, Leonid Levin

NP-Complete vs *NP-Hard*

- A problem p is *NP-Complete* if
  - there is a polynomial-time reduction from **<u>every</u>** problem in *NP* to p.
  - p ∈ *NP*
- A problem p is *NP-Hard* if
  - there is a polynomial-time reduction from **<u>every</u>** problem in *NP* to p.

The SAT Problem: an example

- Consider the boolean expression:

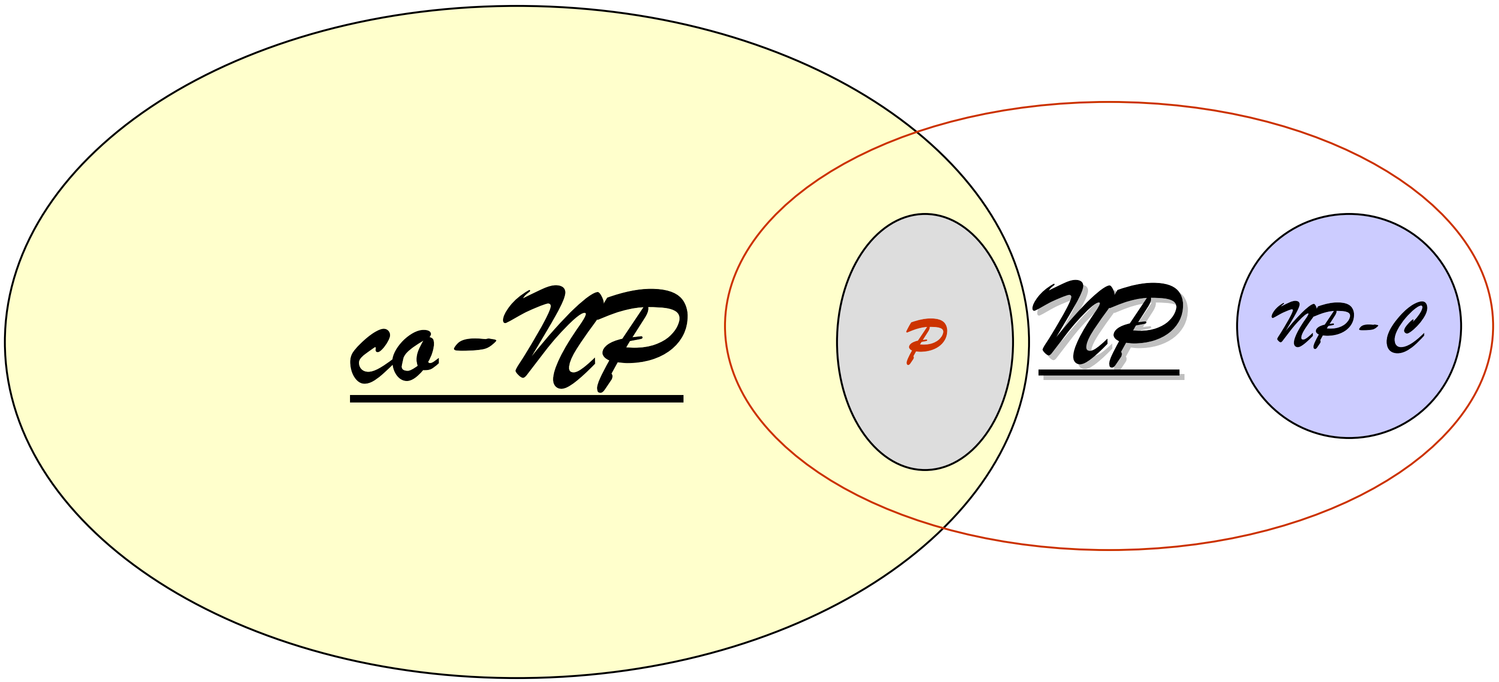
  $C = (a \vee \neg b \vee c) \wedge (\neg a \vee d \vee \neg e) \wedge (a \vee \neg d \vee \neg c)$
- Is C satisfiable?
- Does there exist a True/False assignments to the boolean variables a, b, c, d, e, such that C is True?
- Set a = True and d = True. The others can be set arbitrarily, and C will be true.
- If C has 40,000 variables and 4 million clauses, then it becomes hard to test this.
- If there are n boolean variables, then there are 2^n different truth value assignments.
- However, a solution can be quickly verified!

The SAT (Satisfiability) Problem

- Input: Boolean expression C in Conjunctive normal form (CNF) in n variables and m clauses.
- Question: Is C satisfiable?
  - Let $C = C_1 \wedge C_2 \wedge \ldots \wedge C_m$
  - Where each $C_i = \left(y_1^i \vee y_2^i \vee \cdots \vee y_{k_i}^i\right)$
  - And each $y_j^i \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\}$
  - We want to know if there exists a truth assignment to all the variables in the boolean expression C that makes it true.
- Steve Cook showed that the problem of deciding whether a non-deterministic Turing machine T accepts an input w or not can be written as a boolean expression C_T for a SAT problem. The boolean expression will have length bounded by a polynomial in the size of T and w.

- How to now prove Cook's theorem? Is SAT in $\mathcal{NP}$?
- Can every problem in $\mathcal{NP}$ be poly. reduced to it ?

The problem classes and their relationships



More *NP-Complete* problems

3SAT

- Input: Boolean expression C in Conjunctive normal form (CNF) in n variables and m clauses. Each clause has at most three literals.
- Question: Is C satisfiable?
  - Let $C = C_1 \wedge C_2 \wedge \ldots \wedge C_m$
  - Where each $C_i = \left(y_1^i \vee y_2^i \vee y_3^i\right)$
  - And each $y_j^i \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\}$
  - We want to know if there exists a truth assignment to all the variables in the boolean expression C that makes it true.

3SAT is *NP-Complete.*

More *NP-Complete* problems?

2SAT

- **Input**: Boolean expression C in Conjunctive normal form (CNF) in n variables and m clauses. Each clause has at most <u>three</u> literals.
- **Question**: Is C satisfiable?
  - Let $C = C_1 \wedge C_2 \wedge \ldots \wedge C_m$
  - Where each $C_i = \left(y_1^i \vee y_2^i\right)$
  - And each $y_j^i \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\}$
  - We want to know if there exists a truth assignment to all the variables in the boolean expression C that makes it true.

2SAT is in $\mathcal{P}$.

3SAT is *NP-Complete*

- 3SAT is in *NP*.
- SAT can be reduced in polynomial time to 3SAT.
- This implies that every problem in *NP* can be reduced in polynomial time to 3SAT. Therefore, 3SAT is *NP-Complete*.
- So, we have to design an algorithm such that:
- Input: an instance C of SAT
- Output: an instance C' of 3SAT such that satisfiability is retained. In other words, C is satisfiable if and only if C' is satisfiable.

3SAT is *NP-Complete*

- Let C be an instance of SAT with clauses $C_1, C_2, \ldots, C_m$
- Let C_i be a disjunction of $k > 3$ literals.

  $C_i = \quad y_1 \vee y_2 \vee \ldots \vee y_k$
- Rewrite C_i as follows:

  $C'_i = \quad (y_1 \vee y_2 \vee z_1) \wedge$

  $(\neg z_1 \vee y_3 \vee z_2) \wedge$

  $(\neg z_2 \vee y_4 \vee z_3) \wedge$

  $\ldots$

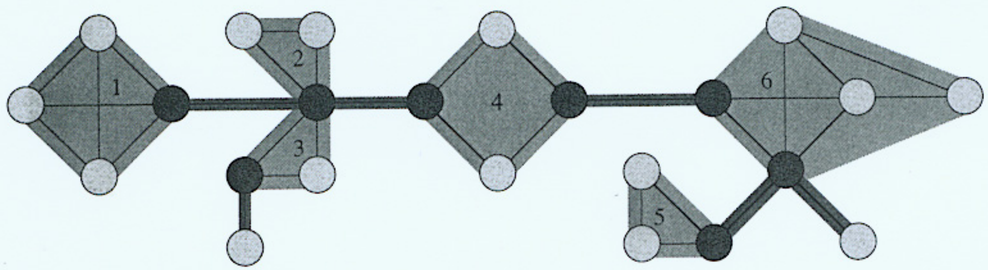
  $(\neg z_{k-3} \vee y_{k-1} \vee y_k)$
- Claim: C_i is satisfiable if and only if C'_i is satisfiable.

2SAT is in $\mathcal{P}$

- If there is only one literal in a clause, it must be set to true.
- If there are two literals in some clause, and if one of them is set to false, then the other must be set to true.
- Using these constraints, it is possible to check if there is some inconsistency.
- How? Homework problem!

The CLIQUE Problem

- A **clique** is a completely connected subgraph.

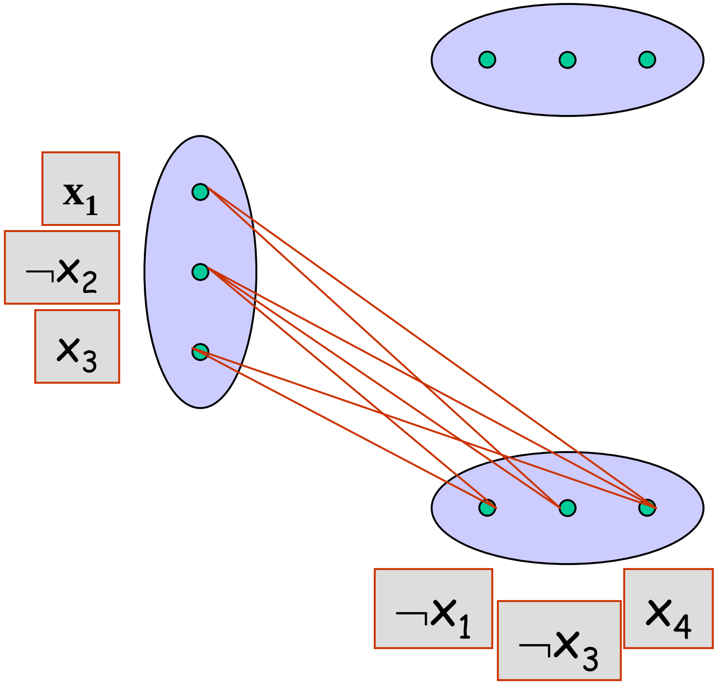


CLIQUE

- Input: Graph G(V,E) and integer k
- Question: Does G have a clique of size k?

CLIQUE is *NP-Complete*

- CLIQUE is in *NP*.
- Reduce 3SAT to CLIQUE in polynomial time.
- $F = (x_1 \vee \neg x_2 \vee x_3)\ (\neg x_1 \vee \neg x_3 \vee x_4)\ (x_2 \vee x_3 \vee \neg x_4)\ (\neg x_1 \vee \neg x_2 \vee x_3)$

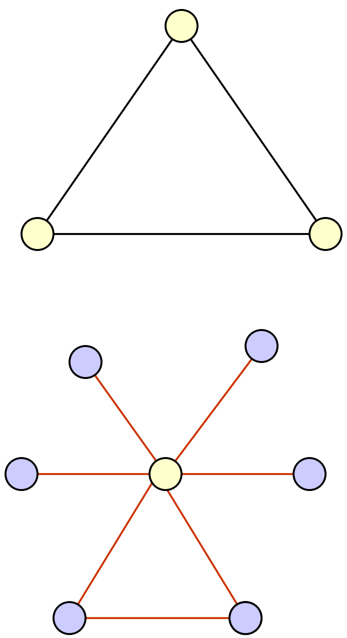


F is satisfiable if and only if G has a clique of size k where k is the number of clauses in F.

Vertex Cover

A vertex cover is a set of vertices that "covers" all the edges of the graph.

Examples



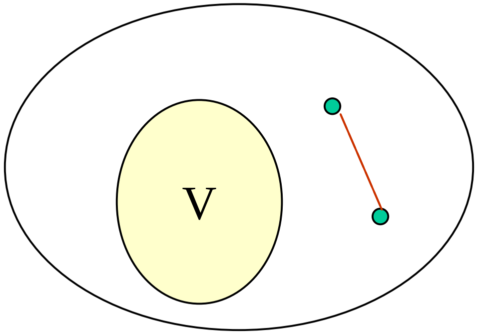
Vertex Cover (VC)

Input: Graph G, integer k

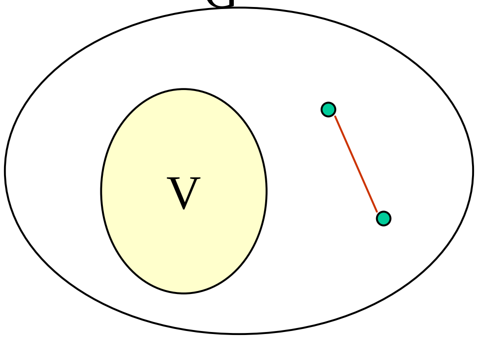
Question: Does G contain a vertex cover of size k?

- VC is in *NP*.
- polynomial-time reduction from CLIQUE to VC.
- Thus VC is *NP-Complete.*

G



G’



Claim: G′ has a clique of size k′ if and only if G has a VC of size k = n – k′

Hamiltonian Cycle Problem (HCP)

Input: Graph G

Question: Does G contain a hamiltonian cycle?

- HCP is in *NP*.
- There exists a polynomial-time reduction from 3SAT to HCP.
- Thus HCP is *NP-Complete.*

- Notes/animations by Yi Ge!