COT 6936: Topics in Algorithms

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http://www.cs.fiu.edu/~giri/teach/COT6936_S10.html

https://online.cis.fiu.edu/portal/course/view.php?id=427

Semester Schedule

- Milestones:
  - By Jan 18: Meet with me and discuss project
  - By Jan 25: Send me email with project team information and topic
  - Feb 3rd week: Short presentation (15 minutes) giving intro to project, problem definition, notation, and background
  - March 2nd week: Take-home Exam
  - Starting March last week: Full length presentation of project (1 hour)
  - April 15: Written report on project

Problems from last lecture

- Achieving diversity in heights:
  - Largest empty range problem
  - Smallest empty range problem
  - Which is harder and why?
- Binary Counter
  - How many bits were changed when a binary counter is incremented from 0 to N?
- Drunken Sailors problem
  - How many sailors will sleep in their own cabins?
- Homework: Robot Challenge problem

NP-Completeness

- Computers and Intractability: A Guide to the theory of NP-Completeness, by Garey and Johnson
  - Compendium (100 pages) of NP-Complete and related problems

Polynomial-time computations

- An algorithm has (worst-case) time complexity O(T(n)) if it runs in time at most cT(n) for some c > 0 and for every input of length n. [Time complexity ≈ worst-case.]
- An algorithm is a polynomial-time algorithm if its (worst-case) time complexity is O(p(n)), where p(n) is some polynomial in n. [Polynomial in what?]
- Composition of polynomials is a polynomial. [What are the implications?]

The class $\mathcal{P}$

- A problem is in $\mathcal{P}$ if there exists a polynomial-time algorithm for the problem. [$\mathcal{P}$ is therefore a class of problems, not algorithms.]
- Examples of $\mathcal{P}$
  - ***DFS:*** Linear-time algorithm exists
  - ***Sorting:*** O(n log n)-time algorithm exists
  - ***Bubble Sort:*** Quadratic-time algorithm $O(n^2)$
  - ***APSP:*** Cubic-time algorithm $O(n^3)$

The class *NP*

- A problem is in *NP* if there exists a non-deterministic polynomial-time algorithm that solves the problem.
- [Alternative definition] A problem is in *NP* if there exists a (deterministic) polynomial-time algorithm that verifies a solution to the problem.
- All problems in *P* are in *NP*. [The converse is the big deal!]

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TSP: Traveling Salesperson Problem

- **Input:**
  - Weighted graph, G
  - Length bound, B
- **Output:**
  - Is there a TSP tour in G of length at most B?
- Is TSP in *NP*?
  - YES. Easy to verify a given solution.
- Is TSP in *P*?
  - OPEN!
  - One of the greatest unsolved problems of this century!
  - Same as asking: **Is *P* = *NP*?**

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So, what is *NP-Complete*?

- *NP-Complete* problems are the "hardest" problems in *NP*.
- We need to formalize the notion of "hardest".

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Terminology

- Problem:
  - An abstract problem is a function (relation) from a set I of instances of the problem to a set S of solutions.
    $p: I \rightarrow S$
  - An instance of a problem p is obtained by assigning values to the parameters of the abstract problem.
  - Thus, describing set of all instances (i.e., possible inputs) and the set of corresponding outputs defines a problem.
- Algorithm:
  - An algorithm that solves problem p must give correct solutions to all instances of the problem.
- Polynomial-time algorithm:

Terminology (Cont'd)

- Input Length:
  - length of an encoding of an instance of the problem.
  - Time and space complexities are written in terms of it.
- Worst-case time/space complexity of an algorithm
  - Is the maximum time/space required by the algorithm on any input of length n.
- Worst-case time/space complexity of a problem
  - UPPER BOUND: worst-case time complexity of best existing algorithm that solves the problem.
  - LOWER BOUND: (provable) worst-case time complexity of best algorithm (need not exist) that could solve the problem.
  - LOWER BOUND ≤ UPPER BOUND
- Complexity Class $\mathcal{P}$:
  - Set of all problems p for which polynomial-time algorithms exist

Terminology (Cont'd)

- Decision Problems:
  - These are problems for which the solution set is {yes, no}
  - Example: Does a given graph have an odd cycle?
  - Example: Does a given weighted graph have a TSP tour of length at most B?
- Complement of a decision problem:
  - These are problems for which the solution is "complemented".
  - Example: Does a given graph NOT have an odd cycle?
  - Example: Is every TSP tour of a given weighted graph of length greater than B?
- Optimization Problems:
  - These are problems where one is maximizing (or minimizing) some objective function.
  - Example: Given a weighted graph, find a MST.
  - Example: Given a weighted graph, find an optimal TSP tour.
- Verification Algorithms:
  - Given a problem instance i and a certificate s, is s a solution for instance i?

Terminology (Cont'd)

- Complexity Class $\mathcal{P}$:
  - Set of all problems p for which polynomial-time algorithms exist.
- Complexity Class $\mathcal{NP}$:
  - Set of all problems p for which polynomial-time verification algorithms exist.
- Complexity Class $co\text{-}\mathcal{NP}$:
  - Set of all problems p for which polynomial-time verification algorithms exist for their **complements**, i.e., their complements are in $\mathcal{NP}$.

Terminology (Cont'd)

- **Reductions:** $p_1 \rightarrow p_2$
  - A problem p_1 is reducible to p_2, if there exists an algorithm R that takes an instance i_1 of p_1 and outputs an instance i_2 of p_2, with the constraint that the solution for i_1 is YES if and only if the solution for i_2 is YES.
  - Thus, R converts YES (NO) instances of p_1 to YES (NO) instances of p_2.
- Polynomial-time reductions: $p_1 \xrightarrow{P} p_2$
  - Reductions that run in polynomial time.

- If $p_1 \xrightarrow{P} p_2$, then
  - If p_2 is easy, then so is p_1. $p_2 \in \mathcal{P} \Rightarrow p_1 \in \mathcal{P}$
  - If p_1 is hard, then so is p_2. $p_1 \notin \mathcal{P} \Rightarrow p_2 \notin \mathcal{P}$

What are $\mathcal{NP}$-*Complete* problems?

- These are the hardest problems in $\mathcal{NP}$.
- A problem p is $\mathcal{NP}$-*Complete* if
  - there is a polynomial-time reduction from **every** problem in $\mathcal{NP}$ to p.
  - $p \in \mathcal{NP}$
- How to prove that a problem is $\mathcal{NP}$-*Complete*?

- Cook's Theorem: [1972]
  - The SAT problem is $\mathcal{NP}$-*Complete*.

Steve Cook, Richard Karp, Leonid Levin

NP-Complete vs *NP-Hard*

- A problem p is *NP-Complete* if
  - there is a polynomial-time reduction from **every** problem in *NP* to p.
  - $p \in NP$
- A problem p is *NP-Hard* if
  - there is a polynomial-time reduction from **every** problem in *NP* to p.
- Remember: to prove problem p is *NP-Complete* you have to reduce a *NP-Complete* problem to p.

The SAT Problem: an example

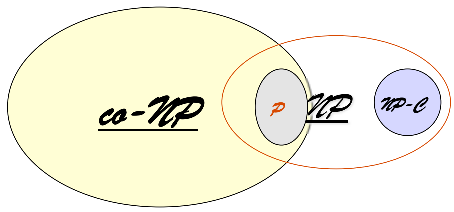
- Consider the boolean expression:
  $C = (a \vee \neg b \vee c) \wedge (\neg a \vee d \vee \neg e) \wedge (a \vee \neg d \vee \neg c)$
- Is C satisfiable? [Does there exist a True/False assignments to the boolean variables a, b, c, d, e, such that C is True?]
- If there are n boolean variables, then there are 2^n different truth value assignments.
- However, a solution can be quickly verified!

The SAT (Satisfiability) Problem

- Input: Boolean expression C in Conjunctive normal form (CNF) in n variables and m clauses.
- Question: Is C satisfiable?
  - Let $C = C_1 \wedge C_2 \wedge \ldots \wedge C_m$
  - Where each $C_i = (y_1^i \vee y_2^i \vee \cdots \vee y_{k_i}^i)$
  - And each $y_j^i \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\}$
  - We want to know if there exists a truth assignment to all the variables in the boolean expression C that makes it true.
- Steve Cook showed that the problem of deciding whether a non-deterministic Turing machine T accepts an input w or not can be written as a boolean expression C_T for a SAT problem. The boolean expression will have length bounded by a polynomial in the size of T and w.

> • How to now prove Cook's theorem? Is SAT in *NP*?
>
> • Can every problem in *NP* be poly. reduced to it ?

The problem classes and their relationships



More NP-Complete problems

3SAT

- Input: Boolean expression C in Conjunctive normal form (CNF) in n variables and m clauses. Each clause has at most three literals.
- Question: Is C satisfiable?
  - Let $C = C_1 \wedge C_2 \wedge \dots \wedge C_m$
  - Where each $C_i = (y_1^i \vee y_2^i \vee y_3^i)$
  - And each $y_j^i \in \{x_1, \neg x_1, x_2, \neg x_2, \dots, x_n, \neg x_n\}$
  - We want to know if there exists a truth assignment to all the variables in the boolean expression C that makes it true.

3SAT is NP-Complete.

3SAT is NP-Complete

- 3SAT is in NP.
- SAT can be reduced in polynomial time to 3SAT.
- This implies that every problem in NP can be reduced in polynomial time to 3SAT. Therefore, 3SAT is NP-Complete.
- So, we have to design an algorithm such that:
- Input: an instance C of SAT
- Output: an instance C' of 3SAT such that satisfiability is retained. In other words, C is satisfiable if and only if C' is satisfiable.

3SAT is NP-Complete

- Let C be an instance of SAT with clauses $C_1, C_2, \ldots, C_m$
- Let C_i be a disjunction of $k > 3$ literals.
  $C_i = y_1 \vee y_2 \vee \ldots \vee y_k$
- Rewrite C_i as follows:
  $C'_i = (y_1 \vee y_2 \vee z_1) \wedge$
  $(\neg z_1 \vee y_3 \vee z_2) \wedge$
  $(\neg z_2 \vee y_4 \vee z_3) \wedge$
  …
  $(\neg z_{k-3} \vee y_{k-1} \vee y_k)$
- Claim: C_i is satisfiable if and only if C'_i is satisfiable.

More NP-Complete problems?

2SAT

- Input: Boolean expression C in Conjunctive normal form (CNF) in n variables and m clauses. Each clause has at most <u>three</u> literals.
- Question: Is C satisfiable?
  - Let $C = C_1 \wedge C_2 \wedge \ldots \wedge C_m$
  - Where each $C_i = (y_1^i \vee y_2^i)$
  - And each $y_j^i \in \{x_1, \neg x_1, x_2, \neg x_2, \ldots, x_n, \neg x_n\}$
  - We want to know if there exists a truth assignment to all the variables in the boolean expression C that makes it true.

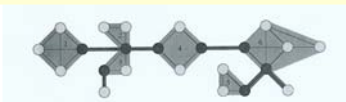
2SAT is in P

2SAT is in P

- If there is only one literal in a clause, it must be set to true.
- If there are two literals in some clause, and if one of them is set to false, then the other must be set to true.
- Using these constraints, it is possible to check if there is some inconsistency.
- How? Homework: do not submit!

The CLIQUE Problem

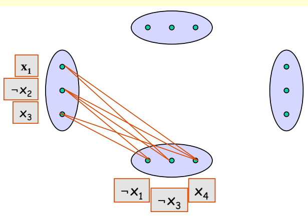
- A **clique** is a completely connected subgraph.



CLIQUE
- Input: Graph G(V,E) and integer k
- Question: Does G have a clique of size k?

CLIQUE is *NP-Complete*

- CLIQUE is in *NP*.
- Reduce 3SAT to CLIQUE in polynomial time.
- $F = (x_1 \vee \neg x_2 \vee x_3)\ (\neg x_1 \vee \neg x_3 \vee x_4)\ (x_2 \vee x_3 \vee \neg x_4)\ (\neg x_1 \vee \neg x_2 \vee x_3)$

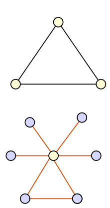


F is satisfiable if and only if G has a clique of size k where k is the number of clauses in F.

Vertex Cover

A vertex cover is a set of vertices that "covers" all the edges of the graph.

Examples

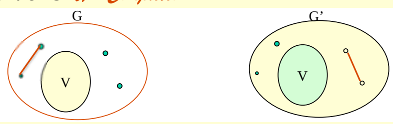


Vertex Cover (VC)

Input: Graph G, integer k

Question: Does G contain a vertex cover of size k?

- VC is in *NP*.
- polynomial-time reduction from CLIQUE to VC.
- Thus VC is *NP-Complete.*



Claim: G' has a clique of size k' if and only if G has a VC of size k = n – k'

Hamiltonian Cycle Problem (HCP)

Input: Graph G

Question: Does G contain a hamiltonian cycle?

- HCP is in *NP*.
- There exists a polynomial-time reduction from 3SAT to HCP.
- Thus HCP is *NP-Complete.*

Shortest Path vs Longest Path

Input: Graph G with edge weights, vertices u and v, bound B

Question: Does G contain a shortest path from u to v of length at most B?

Question: Does G contain a longest path from u to v of length at most B?

Homework: Listen to Cool MP3:
http://www.cs.princeton.edu/~wayne/kleinberg-tardos/longest-path.mp3

Perfect (2-D) Matching vs 3-D Matching

1. Input: Bipartite graph, G(U,V,E)
   Question: Does G have a perfect matching?
2. Input: Sets U and V, and E = subset of U×V
   Question: Is there a subset of E of size |U| that covers U and V? [Related to 1.]
3. Input: Sets U,V,W, & E = subset of U×V×W
   Question: Is there a subset of E of size |U| that covers U, V and W?

Coping with NP-Completeness

- Approximation: Search for an "almost" optimal solution with provable quality.
- Randomization: Design algorithms that find "provably" good solutions with high prob and/or run fast on the average.
- Restrict the inputs (e.g., planar graphs), or fix some input parameters.
- Heuristics: Design algorithms that work "reasonably well".