ELSEVIER Theoretical Computer Science 140 (1995) 249-263

Theoretical Computer Science

A note on the combinatorial structure of the visibility graph in simple polygons ☆

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Abstract

Combinatorial structure of visibility is probably one of the most fascinating and interesting areas of engineering and computer science. The usefulness of visibility graphs in computational geometry and robotic navigation problems like motion planning, unknown-terrain learning, shortest-path planning, etc., cannot be overstressed. The visibility graph, apart from being an important data structure for storing and updating geometric information, is a valuable mathematical tool in probing and understanding the nature of shapes of polygonal and polyhedral objects. In this research we wish to initially focus our attention on a fundamental class of geometric objects. These geometric objects may be looked upon as building blocks for more complex geometric objects, and which offer an ideal balance between complexity and simplicity, namely simple polygons.

A major theme of the proposed paper is the investigation of the combinatorial structure of the visibility graph. More importantly, the goals of this paper are:

(i) To characterize the visibility graphs of simple polygons by obtaining necessary and sufficient conditions a graph must satisfy to qualify for the visibility graph of a simple polygon

(ii) To obtain hierarchical relationships between visibility graphs of simple polygons of a given number of vertices by treating them as representing simple polygons that are deformations of one another.

(iii) To exploit the potential of complete graphs to be natural coordinate systems for addressing the problem of reconstructing a simple polygon from visibility graph. We intend to achieve this by defining appropriate "betweenness" relationships on points with respect to the edges of the complete graphs.

1. Introduction

Visibility is an important concept in computational geometry, and is crucial in obtaining optimal solutions to many geometric problems. Visibility problems occupy an important place in robotic navigation and scene analysis, graphics, and related areas. For instance, obtaining shortest paths between given pairs of points in regions

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☆ This research is funded by the Office of Naval Research.

Elsevier Science B.V.
SSDI 0304-3975(94)00170-7

populated with polygonal or polyhedral obstacles is a typical problem in robotic navigation. Its solution often draws heavily from the visibility relationships among the vertices and edges of the obstacles in the domain under consideration. For a broader treatment on the application of visibility graph structure, see [10, 14, 16]. Visibility representations of planar graph constructed by mapping vertices to horizontal segments and edges to vertical segments have been studied in detail by Tamassia and Tollis [15]. Recently, Guibas et al. [8] presented an efficient solution to the robot localization problem in two dimensions.

1.1. Motivation: relevance of characterization of visibility graphs for many interesting applications

Aside from being an interesting and fundamental computational geometric problem, the problem has several applications in robot navigation, scene analysis, pattern recognition, and other related areas [1, 2].

The solution of the problem of characterization of visibility graphs of simple polygons is of considerable theoretical value in computational geometry and graph theory, for it gives a method of qualification and concrete representation of shapes in geometry, and recognizes an entire class of interesting new graphs hitherto unknown in graph theory. However, the characterization problem invariably raises the associated problem of reconstruction of a simple polygon from its visibility graph. Solution of the reconstruction problem is of great value to various applications in robotics and pattern recognition. For example, in robotics, it is often convenient to encode information about shapes of the terrain areas to be traversed by the robot in the form of incidence matrices representing graphs, if possible, for the purposes of storage and communication. But in order to utilize this information, it is necessary to decode it and obtain the shape of the terrain area. This is very useful in computing optimal paths and in collective and sympathetic learning of an unknown terrain by a group of intercommunicating robots exploring the terrain.

In pattern recognition too, the characterization and reconstruction problems' solutions would have important and far-reaching applications, for, to compare two simple polygonal shapes, one can compare their visibility graphs to obtain a measure of their similarity/disparity. The main advantage of using visibility graphs in shape recognition is the comparison independent of the relative scale or orientation of the objects compared! In this proposal, we address the problem of characterization of visibility graphs of a restricted class of simple polygons, and the reconstruction of these polygons from their visibility graphs. These visibility graphs represent *generic* shapes in that all polygons with the same visibility graph have *recognizably* the same shape even if they are not similar or congruent in the strong geometric sense. This is an extremely important property, and helps compare and identify objects of similar shapes even though they may be represented at different scales or have different relative orientations, by decomposing the objects into smaller simple polygons, and obtaining their visibility graphs. These important applications of the problems of characterization and reconstruction are the chief motivations for our attempt to solve these problems. We believe that obtaining an efficient algorithmic solution to the reconstruction problem is closely related to the understanding and solution of the characterization problem for an appropriate class of simple polygons.

1.2. Background

Visibility in its simplest form is a straightforward concept. Two points in space are *mutually visible* or *constitute a visible pair* iff the line segment joining them does not intersect any geometric object other than those points. There are many variants of this basic concept motivated by various applications (for example, see [11]). Associated closely with the concept of visibility is a data structure called the visibility graph, which stores the visibility relationships among a given set of points. Given a set of n points in space, one can construct the visibility graph of this set of points by considering a set of n vertices, each vertex representing a given point in space, and joining only those pairs of vertices representing pairs of visible points in space by edges.

Extensive work has been done in the development and utilization of the visibility graph as a computational tool, and while algorithms have been developed to compute visibility graphs of a set of polygonal obstacles in the plane (for example, by Hershberger [9] and Ghosh and Mount [7]), visibility graphs themselves have eluded a categorical description or classification from the graph-theoretic point of view. In 1985 ElGindy [3] showed that every maximal outerplanar graph is the vertex visibility graph of a monotone polygon. However, in general, visibility graphs have no known combinatorial structure or characterization, and they do not belong to any known category of graphs in graph theory. Nevertheless, an understanding of their structure and properties promises invaluable insights into the nature of shapes of polygons and polyhedra, and the "*distribution of space*" within their boundaries, not to mention solutions to important long-standing problems.

Although the body of work done in the area of visibility graphs and motion planning is extensive, the work concerning "computational characterization of visibility graphs of simple polygons" is still in its infancy. Indeed, as *O'Rourke remarked* in his book [11]:

> "It is my belief that some of the fundamental unsolved problems involving visibility in computational geometry will not be solved until the combinatorial structure of visibility is more fully understood. Perhaps the purest condensation of this structure is a *visibility graph*."

In this proposal, we will concentrate on computational characterization of visibility graphs of simple polygons.

2. Scope of the paper

We wish to investigate the characteristics of visibility graphs of certain simple but fundamental geometric objects in the plane, namely simple polygons. Simple polygons may be thought of as building blocks to construct more complex polygonal regions in the plane. Decomposition of a complex polygonal region into simple polygons, apart from being less tedious than decomposition into convex or star-like polygons, or triangulation, also distributes the complexity of the shape of the parent object into smaller, simpler and more manageable objects. We believe that understanding the visibility graphs of simple polygons and characterizing them by a few properties is both possible and fruitful. We wish to address in particular the following problems:

(i) Characterization of visibility graphs of simple polygons by obtaining necessary and sufficient conditions to be satisfied by a graph to qualify for the visibility graph of a simple polygon.

(ii) Obtaining relationships between visibility graphs with the same number of vertices (by treating the visibility graphs as representations of simple polygons that are deformations of one another).

(iii) Constructing a simple polygon from a visibility graph.

The development of our characterization of visibility graph of a simple polygon (which has been an open problem) involves the construction of a high performance data structure for the characterization-sensitive properties of the visibility graphs.

Ghosh [5] obtained necessary conditions for a graph with a Hamiltonian cycle to be the visibility graph of a simple polygon. We believe that these necessary conditions are also sufficient for a restricted class of visibility graphs of simple polygons, these simple polygons being such that all simple polygons are naturally decomposable into them. It is our goal to characterize this restricted class of visibility graphs.

3. Organization of the paper

In the next section, we give the necessary definitions and explain them with examples. Section 5 will contain a description of the previous works and their limitations. Section 6 outlines our approach. In Section 7, the significance of our approach is discussed.

4. Definitions and examples

Definition. In the plane a polygon is a finite set of straight line segments with the property that every segment extreme is shared by exactly two segments of the polygon, and no proper subset of the set of segments has the same property. The segments are the edges and their extremes are the vertices of the polygon.

Definition. A simple polygon is a polygon whose no two nonconsecutive edges share a point.

A simple polygon divides the plane into exactly two disjoint regions, an interior and an exterior. For example, in Fig. 1, (i) is a simple polygon, whereas (ii) is not.

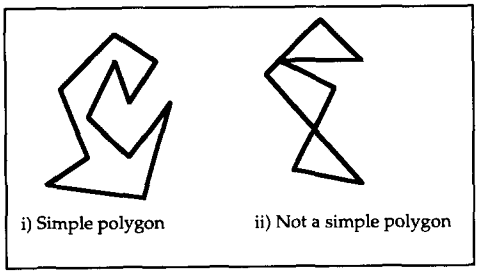

Fig. 1. Polygons.

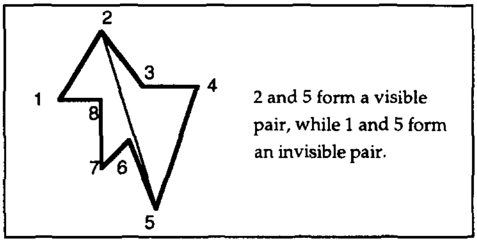

Fig. 2. Visibility properties.

Hereafter, when we refer to a simple polygon, we refer to the polygon *with* its interior included. The edges and vertices of the simple polygon will be referred to as its *boundary*.

Definition. Let P be a simple polygon. Two vertices u and v of P *are visible to each other*, or *form a visible pair*, iff the line segment $\underline{uv}$ joining the vertices u and v does not intersect the exterior of P. A pair of vertices not visible to each other is called an *invisible pair* (see Fig. 2).

The visibility graph of a simple polygon with n vertices is obtained by constructing a graph on n vertices, each vertex of the graph representing a vertex of the simple polygon, and each edge of the graph joining only those pairs of vertices that represent visible pairs of vertices in the polygon P. Since adjacent vertices of P are visible to each other, by our definition of visibility, the visibility graph of P has a Hamiltonian cycle. We adopt the convention of representing the visibility graph on n vertices by a straight line graph G in the plane, with the Hamiltonian cycle forming the boundary of a convex polygon as shown in Fig. 3.

The visibility graph of a convex polygon with n vertices is the complete graph K^n on n vertices, since, in a convex polygon, every vertex is visible to every other vertex.

Definition. If $v_i, v_{i+1}, \ldots, v_j$ is a contiguous subsequence of the sequence $v_1, \ldots, v_n$ of vertices of the Hamiltonian cycle traversed in clockwise order, then the subsequence $v_i, v_{i+1}, \ldots, v_j$ is called a closed chain, denoted $[v_i, \ldots, v_j]$. If either one of the vertices v_i

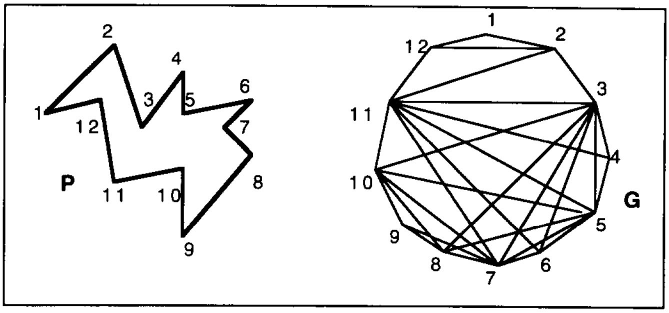

Fig. 3. Hamiltonian cycle forming the boundary of a convex polygon.

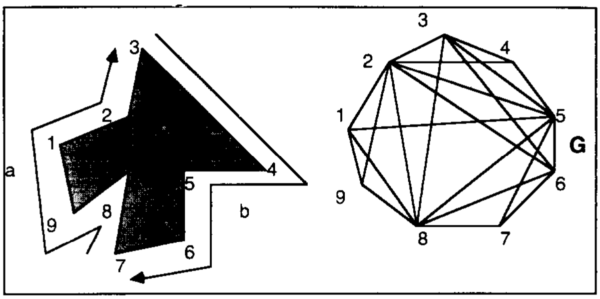

Fig. 4. Visibility graph.

or v_j were to be omitted from the subsequence, then we would have the half-open chains $(v_i, \ldots, v_j]$ or $[v_i, \ldots, v_j)$, as the case may be. In the event of both the extreme vertices v_i and v_j being omitted, we would have the open chain $(v_i, \ldots, v_j)$ (see Fig. 4).

In the visibility graph in Fig. 4 G of P the closed chains $a = [7, \ldots, 3]$ and $b = [3, \ldots, 7]$ are shown by crooked arrows.

Definition. If $v_{i_1}, \ldots, v_{i_k}$ is a subsequence of the sequence $v_1, \ldots, v_n$ of the Hamiltonian cycle such that $(v_{i_1}, v_{i_2}), (v_{i_2}, v_{i_3}), \ldots, (v_{i_{k-1}}, v_{i_k}), (v_{i_k}, v_{i_1})$ are edges in G, then the vertices $v_{i_1}, \ldots, v_{i_k}$ are said to form a subcycle $\langle v_{i_1}, \ldots, v_{i_k} \rangle$ of the Hamiltonian cycle $\langle v_1, \ldots, v_n \rangle$.

5. Summary and critical review of previous work

ElGindy and Avis [4] posed the problem of determining whether a graph with a given Hamiltonian cycle can be embedded in the plane as a visibility graph of a simple polygon with the Hamiltonian cycle forming the boundary of the polygon. ElGindy [3] succeeded in solving this problem for maximal outerplanar graphs. The problem of obtaining necessary and sufficient conditions that a graph must satisfy in order to qualify for the visibility graph of a simple polygon was addressed by Ghosh [5]. He obtained three necessary conditions a graph with Hamiltonian cycle must satisfy to qualify for the visibility graph of a simple polygon, and conjectured that these conditions were also sufficient. We briefly discuss the main ideas of his work here.

Let P be a simple polygon with n vertices numbered 0 through $n-1$ in clockwise order (say). Since neighbouring vertices are visible to each other, the visibility graph G of P will contain the (undirected) edges $(v_0, v_1), \ldots, (v_{n-2}, v_{n-1}), (v_{n-1}, v_0)$, forming a Hamiltonian cycle. Thus (v_i, v_j) is an edge in G iff v_i and v_j are visible to each other in P. Vertices v_i and v_j are invisible to each other in P iff their line-of-sight is obstructed by edges on the boundary of P. In other words, if any vertex v_k belonging to the open chain $(v_i, \ldots, v_j)$ or $(v_j, \ldots, v_i)$ lies to the right of the directed line segment v_iv_j or v_jv_i, respectively, then v_i and v_j are invisible to each other. v_k then can be looked upon as a *blocking vertex* of the invisible pair (v_i, v_j). This idea of a blocking vertex, however, is geometric and does not have a direct visibility graph counterpart. Ghosh uses the following definition of a blocking vertex.

Definition. A vertex v_k is a *blocking vertex* for the invisible pair (v_i, v_j) if no vertex of the half-open chain $[v_i, \ldots, v_k)$ is visible to a vertex of the half-open chain $(v_k, \ldots, v_j]$.

Definition. A vertex v_k is a minimal blocking vertex of the invisible pair of (v_i, v_j) if it is a blocking vertex for (v_i,v_j), and, further, (v_i, v_k) and (v_k, v_j) are visible pairs.

If (v_i, v_j) is a visible pair in G, then the Hamiltonian cycle $\langle v_1, \ldots, v_n \rangle$ is split into two subcycles by the edge (v_i, v_j), for assume (w.l.o.g.) that $i<j$, then the subcycles into which $\langle v_i, \ldots, v_n \rangle$ is split by (v_i, v_j) are $\langle v_i, v_{i+1}, \ldots, v_{j-1}, v_j \rangle$ and $\langle v_j, v_{j+1}, \ldots, v_n, v_1, \ldots, v_{i-1}, v_i \rangle$.

Remark. If (v_k, v_1) is the invisible pair in the open chain $(v_i, \ldots, v_j)$, then it cannot have a blocking vertex belonging to the open chain $(v_j, \ldots, v_i)$.

Definition. If C is a subcycle of the Hamiltonian cycle of the visibility graph of a simple polygon P, then the restriction of G to the subcycle C induces a subgraph G', whose vertex set V' is the set of vertices in C and whose edge set E' is the restriction of the edge set E of G to those edges between vertices in V'. G' is called the graph induced by the subcycle C (see Fig. 5).

5.1. Ghosh's [5] contribution a critical review

Ghosh put forward the following three necessary conditions on a graph G with a Hamiltonian cycle to qualify for the visibility graph of a simple polygon P:

(1) Every invisible pair (v_i, v_j) in G has a blocking vertex v_k.

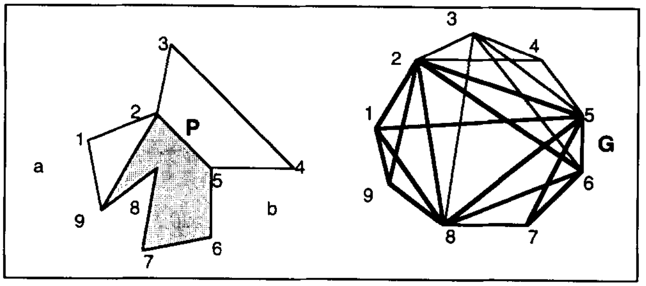

Fig. 5. The subcycle $\langle 1,2,5,6,7,8,9 \rangle$ induces the visibility subgraph of G, indicated by the thickened edges. The corresponding subpolygon of P is indicated by the shaded area.

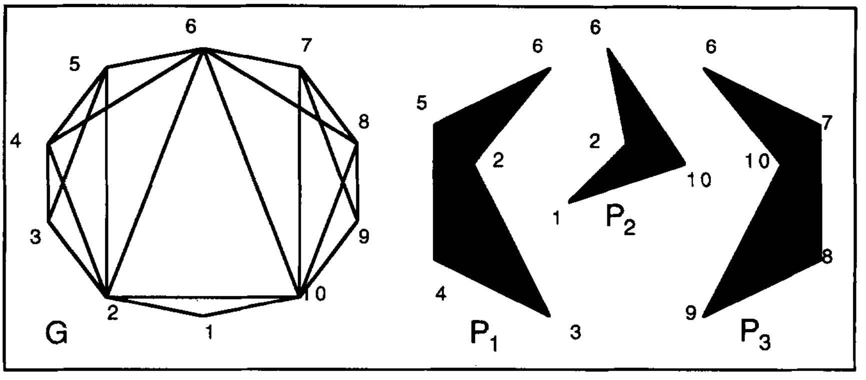

Fig. 6. An example which shows that the conditions are not sufficient.

(2) If C is any subcycle of the Hamiltonian cycle, then the edge set E', of the subgraph G' induced by C, contains a triangulation of the cycle C.

(3) If v_i, v_j, v_k, v_l and v_m occur in that order on the Hamiltonian cycle of G, and if (v_i, v_j) and (v_l, v_m) are invisible pairs, then v_k cannot be only blocking vertex of both the invisible pairs at the same time.

It is not hard to see that the above three conditions are necessary for a graph to be the visibility graph of a simple polygon. We shall therefore omit the proofs for the sake of brevity (for details see [5]).

Ghosh conjectured that these conditions were also sufficient for a graph to satisfy in order to be the visibility graph of a simple polygon. However, the example in Fig. 6 shows that the conditions are not sufficient. The graph G in the figure satisfies the three necessary conditions as can be easily verified; yet it cannot be the visibility graph of any simple polygon. Indeed the subcycles $\langle 2,3,4,5,6 \rangle$ and $\langle 6,7,8,9,10 \rangle$ induce visibility graphs of pentagons P_1 and P_3 with 2 and 10 as reflex vertices, respectively, since 2 is the blocking vertex for the invisible pair (3, 6), and 10 is the blocking vertex of the invisible pair (6, 9). The subcycle $\langle 1,2,6,10 \rangle$ induces the visibility graph of a quadrilateral P_2, with either 2 or 10 as a reflex vertex, since (6, 1) has 2 and 10 as its blocking vertices. However, not both 2 and 10 can at the same time be reflex; but the pentagons P_1 and P_3 require both 2 and 10 to be reflex, which is impossible. In other words, if we try to glue together the pentagons and the quadrilateral along their common edges, the quadrilateral would overlap with one of the pentagons. Hence G cannot be the visibility graph of a simple polygon.

Thus there are either more than three conditions constituting the set of necessary and sufficient conditions for a graph to be the visibility graph of a simple polygon, or the above three conditions are necessary and sufficient for a more restricted class of graphs to be the visibility graphs to be the visibility graphs of simple polygons.

In the work summarized above, we notice the following shortcomings:

(i) The absence of a rigorous framework or a computational model for addressing the problem.

(ii) The definition of a blocking vertex in a visibility graph is looser and more general, and hence does not exactly correspond to the definition of a blocking vertex of a polygon.

(iii) There is no strategy for verifying whether a given set of necessary conditions is also sufficient.

(iv) The absence of a constructive approach to facilitate reconstructions of polygons from visibility graphs, and design of algorithms in this direction.

6. Our approach

Our approach to the characterization problem promises to remedy these.

6.1. *Irreducibility*

As was demonstrated by the counterexample of the previous section, the necessary conditions proposed by Ghosh are not sufficient for a graph with a known Hamiltonian cycle to be the visibility graph of a simple polygon. However, we conjecture that the three necessary conditions mentioned above are also sufficient for a more restricted category of graphs, which we propose to call irreducible graphs. A few definitions and remarks are in order here.

Definition. If v_i, v_j, v_k and v_l are vertices located cyclically on Hamiltonian cycle of a graph G, and if (v_i, v_k) and (v_j, v_l) are edges in G, then the visible pairs (v_i, v_k) and (v_j, v_l) are said to be cross-visible to each other (see Fig. 7).

Remark. If G is the visibility graph of a simple polygon P, and (v_i, v_j) is a visible pair such that no other visible pair is cross-visible to it, then it follows that no vertex in the open chain $(v_i, \ldots, v_j)$ is visible to any vertex in the open chain $(v_j, \ldots, v_i)$.

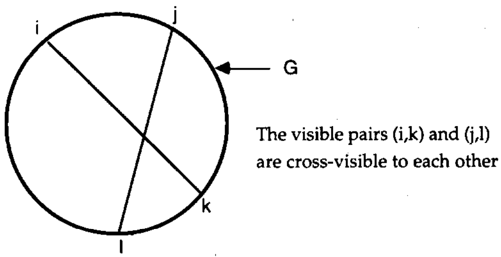

Fig. 7. Visible pairs.

The above remark suggests that the two subcycles $C' = \langle v_i, \ldots, v_j \rangle$ and $C'' = \langle v_j, \ldots, v_i \rangle$ induce visibility subgraphs G' and G'', which are in a sense independent of each other. In other words, the graphs G' and G'' share no visibility relationships and, in this sense, are divorced from each other. We may then go on to look upon them as visibility graphs in their own right, and recover polygons P' and P'' corresponding to them. We may, then, later on obtain a method of deforming P' and P'' so as not to disturb the visibility relationships of their vertices, but at the same time obtain on "gluing" them together along their common edge (v_i, v_j) a polygon Q whose visibility graph is the same as that of P.

The above operation of splitting a visibility graph G along a visible pair to which no other visible pair is cross-visible, and obtaining visibility graphs G' and G'', suggests that we may repeatedly apply this process until we obtain a collection of visibility subgraphs $G_1, G_2, \ldots, G_k$ of G such that none of these graphs has a visible pair which is not cross-visible to any other visible pair. This prompts the following definition.

Definition. If G is a graph such that every visible pair of G is cross-visible to some other visible pair of G, then G is said to be an irreducible graph.

The following lemma is immediate.

Lemma. *Any visibility graph can be decomposed in a unique way into irreducible visibility subgraphs.*

We cannot resist the temptation of drawing an analogy between the role of prime numbers among natural numbers, and that of irreducible visibility graphs among visibility graphs of simple polygons.

Irreducibility introduces a sort of rigidity to a visibility graph, and the shape of the corresponding polygon is more constrained and less pliable, insofar as deformations which leave its visibility graph invariant are concerned. In other words, the *generic* shape of the simple polygons of an irreducible visibility graph is unique. We note at this point that the graph in the counterexample above has no cross-visibility across its visible pair (2, 10). Indeed, if we try to introduce cross-visibility across pair (2, 10), it is easy to see that necessary condition 3 is violated.

6.2. A new conjecture

This leads us to conjecture the following:

The necessary conditions of a Ghosh are also sufficient for any irreducible graph to satisfy in order to be the visibility graph of a simple polygon.

It is our goal to investigate this conjecture by either proving it or by refuting it. In the event of the conjecture being false, we would like to investigate the possibility of formulating an alternate set of necessary conditions and proving their sufficiency. Obtaining the "correct" set of necessary conditions is one thing, and proving the sufficiency is another matter. We now discuss our proof strategy for showing that a set of necessary conditions is also sufficient.

6.3. Proof strategy: an outline

Just proving the sufficiency of a set of necessary conditions may not mean much, unless an insight into the interrelationships of visibility graphs and some relative arrangement of them is obtained. We will state two theorems which will not only help prove the sufficiency but will also provide valuable insight into the hierarchical nature of visibility graphs.

Let N be the set of necessary conditions for an irreducible graph with a Hamiltonian cycle to be the visibility graph of a simple polygon.

Definition. A graph G with a known Hamiltonian cycle that satisfies the necessary conditions N is called a potential visibility graph (pvg) with respect to N.

1. The inflating theorem. *If G is an irreducible pvg on n vertices which is not the complete graph K^n, then there is an edge in $K^n \setminus G$ such that $G' = G \cup \{e\}$ is also an irreducible pvg.*

This theorem is geometrically equivalent to inflating a simple polygon by pushing out its reflex vertices and creating new visible pairs.

2. The deflating theorem. *If G is an irreducible visibility graph on n vertices, and there exists an edge of G such that $G' = G \setminus \{e\}$ is an irreducible pvg, then G' is also an irreducible visibility graph.*

This theorem is geometrically equivalent to deflating a simple polygon by pushing in its vertices and creating new invisible pairs.

The inflating and deflating theorems help us investigate the sufficiency of a set of necessary conditions as follows.

Let N be the set of necessary conditions (that are also sufficient, say), and G any irreducible graph on n vertices with a known Hamiltonian cycle, which is not the complete graph K^n. Further, let G be a pvg w.r.t. N. Then, by repeatedly applying the inflating theorem, we can find a finite sequence of edges $e_1, \ldots, e_k$ belonging to $G^c = K^n \backslash G$ and a sequence of irreducible pvg's $G_i = G_{i-1} \cup \{e_i\}$, $1 \leqslant i \leqslant k$, with $G_0 = G$ and $G_k = K^n$.

Now K^n is the visibility graph of a convex polygon on n vertices. By applying the deflating theorem to $G_k = K^n$, we find that $G_{k-1} = G_k \backslash \{e_k\}$ is also an irreducible visibility graph. By repeatedly applying the deflating theorem to the graphs $G_{k-1}, \ldots, G_1$ we conclude that $G_{k-1}, \ldots, G_l$, and in particular G_0 are all irreducible visibility graphs. This establishes the result that the irreducible pvg G_0 is indeed the visibility graph of a simple polygon. Hence, N is shown to be also sufficient. If on the other hand the set N of necessary conditions was not sufficient, then either the inflating or the deflating theorem would fail to hold. Thus we have the following theorem.

Theorem. *If N is a set of necessary conditions such that for any irreducible pvg w.r.t N the inflating and the deflating theorems hold, then set N is also a set of sufficient conditions.*

The propose to prove the inflating and deflating theorems for irreducible pvg's with respect to the three necessary conditions of Ghosh.

6.4. Hierarchical structure of visibility graphs

The above strategy for proving the sufficiency of a set of necessary conditions also hints at a natural way to relate visibility graphs.

Let $V_1, \ldots, V_k$ be all possible irreducible visibility graphs on n vertices. Consider the undirected graph $\boldsymbol{G}^n$ with its vertex set given by $V = \{V_1, \ldots, V_k\}$, and its edge set $\boldsymbol{E}$ defined as follows: An edge (V_i, V_j) belongs to $\boldsymbol{E}$ iff V_i can be obtained from V_j by either the addition or deletion of a visible pair. Any path in $\boldsymbol{G}^n$ from a vertex V_i to a vertex V_k would specify a sequence of deformations on a polygon P_i with visibility graph V_i, leading to a polygon P_k with visibility graph V_k. Thus, if N was indeed the set of necessary conditions that are also sufficient, then our proof strategy would correspond to traversing up and down a path connecting V_i and V_k in $\boldsymbol{G}^n$ (see Fig. 8). The arrows ascending from G_0 to $G_k = K^n$ indicate the path taken by repeated application of the inflating theorem on $G_0, \ldots, G_{k-1}$, while the arrows descending from K^n to G_0 indicate the path taken by repeated application of the deflating theorem on $K^n, \ldots, G_1$.

We propose to investigate the structure of the *pedigree* $\boldsymbol{G}^n$ of visibility graphs on n vertices, as we are convinced that this would lead to a profound graph-theoretic

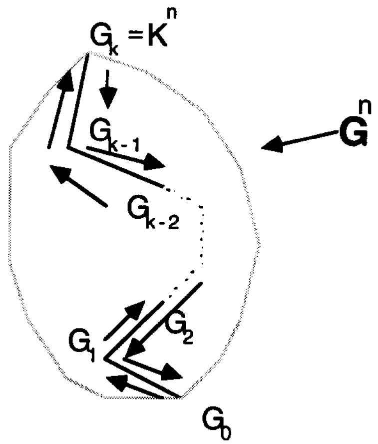

Fig. 8. Hierarchical structure of visibility graphs.

understanding of the structure and spatial properties of simple polygons and their visibility graphs.

7. A technique to construct a simple polygon from a visibility graph with our new approach

An equally interesting (and algorithmically more important) issue is that of the recovery of a polygon from a visibility graph. This raises the question of whether there is a unique polygon associated with a given visibility graph. The answer to this is clearly in the negative, since it is obvious that one may perform certain continuous deformations on a polygon without changing its visibility graph. But, if we restrict ourselves to irreducible visibility graphs of simple polygons, then it seems that though there are infinitely many polygons with a given visibility graph, they all share the same *generic* shape and geometric features of that visibility graph. This points to exciting applications in pattern matching and pattern recognition! The task of constructing a polygon from a visibility graph is nontrivial. This calls for a special computational tool to handle the visibility graph. As mentioned earlier in this paper, we represent the visibility graph of a simple polygon in the plane as a straight line graph, with its Hamiltonian cycle forming the boundary of a convex polygon. We intend to evolve a method of deforming the Hamiltonian cycle to obtain the polygon by satisfying the visibility constraints imposed by the visibility graph's edges. In order to do this systematically and algorithmically, we propose to develop the complete graph K^n on n vertices as a coordinate system for the visibility graph on n vertices, so that the Hamiltonian cycle might be "bent" in the appropriate places to yield a polygon of the given visibility graph. The complete graph K^n is naturally suited for this since one does not need numerical coordinate positions of vertices and edges but only relative spatial arrangements of vertices and edges to meet the desired visibility criteria of the visibility graph. We have several partial and promising results in this direction.

Another issue is that of "gluing" together several polygons corresponding to irreducible visibility graphs obtained by decomposing a visibility graph G in order to obtain the polygon P corresponding to it. This requires that we deform the polygons in such a way as not to change the visibility graphs, but, at the same time, ensure that gluing them together by their commonly shared edges will not introduce cross-visibility with respect to these edges. This requires a geometric "gluing lemma" which we are in the process of formulating in our researches in this direction.

8. Concluding remarks

In this paper, we have discussed the problem of characterization of visibility graphs of simple polygons. We have proposed certain very promising directions in not only attacking this problem, but also the algorithmically more important problem of constructing simple polygons from visibility graphs. The applications of visibility graphs and their generic-shape characterization properties hint at exciting applications to pattern-matching, shape recognition and image processing. There are several interesting ramifications of the questions addressed in this paper that promise important applications in the fields of robotics and computational geometry. We are addressing these problems and have several partial results and valuable insights.

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