INFORMATION SCIENCES **26**, 127–139 (1982)

A Heuristic Algorithm for Optimal Placement of Rectangular Objects

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Communicated by John M. Richardson

ABSTRACT

The problem of partitioning a two-dimensional area into pieces having certain sizes with a minimum of wasted space is very important, especially in packing components tightly in the manufacture of very large-scale integrated circuits. The purpose of this paper is to examine the problem of placing rectangular objects in a rectangular area so as to minimize the wasted space, from the viewpoint of establishing maximum empty rectangles rather than the standard linear-programming approach. A comparison of our results with those of the Steudel [5] is reported. Empirical comparisons of our results indicate that our algorithm is very simple and efficient.

INTRODUCTION

The problem of partitioning a two-dimensional area into pieces having certain sizes with a minimum of wasted space is very important, especially in garment manufacture, metal fabrication, publication layout and, more recently, the packing of components tightly in very large-scale integrated circuits. Several aspects of this problem have been studied in great detail. The "knapsack problem," for example, involves cutting an object into smaller pieces with each piece having a given length and some value associated with it in such a way as to maximize the total value of the pieces so produced. Gilmore and Gomory [2] examined this problem and showed that it was essentially a linear-programming problem. They note, however, that the number of columns will, in higher dimensions, be so large that this approach will be impractical. A more restrictive approach is the "cutting-stock problem," in which only guillotine cuts (i.e., cuts

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going completely from edge to edge) are allowed at any particular instant. This problem was examined by two methods: the tree-search method due to Christofides and Whitlock [1] and a heuristic approach due to Steudel [5].

The problem we shall consider is not contained in either of the above classes, but is important nevertheless. It consists of choosing a subset of rectangular objects from a given set and placing them in a rectangular area so as to minimize the wasted space. The objects are allowed to have any size, although a simple case for examination is one in which they are identical. In linear-programming terms, we associate a shadow price, P_i, with each piece, and then the problem becomes:

Maximize $\Sigma_{i=1}^{N} a_i P_i$, subject to $a_i = 0$ or $a_i = 1$, and so that $\{a_1, a_2, \ldots, a_N\}$ corresponds to fitting each l_i-by-w_i object into the L-by-W area.

There currently exists no economical solution to the problem as stated.

We shall attempt to make several propositions, examine some simple cases, and then propose our heuristic algorithm.

The remainder of this paper is organized as follows: an informal proposal of the method, the heuristic method, a description of the algorithm, discussion of the results, and conclusions.

1. INFORMAL DESCRIPTION OF THE PROPOSED METHOD

We will use the notation of Jayakumar [4], defining the maximum empty rectangles (MERs) as that unique set of possibly overlapping rectangles bounded only by the edges of the area and by the objects within the area (see the Appendix). For a single object, the assignment appears in Figure 1. We will take the area to be initially L by W and let the dimensions of the ith object be l_i by w_i. Also, we will let B be the number of objects placed so far, and R be the number of MERs which currently exist. Initially, there will be no objects in the space and the area itself is a single MER, so that $R = 1$ and $B = 0$.

As noted by Jayakumar [4], R is a function of L and W as well as B through the set $\{(l_i, w_i)\}$, for i between 0 and B. In any given situation, the functional dependence of R upon B is extremely complex, having many local extrema.

We note first that both R and B are bounded. If $l_{\min}$ is the minimum of the set $\{l_i\}$ and $w_{\min}$ that of $\{w_i\}$, then

$$B_{\max} = \text{floor}(L/l_{\min}) \times \text{floor}(W/w_{\min}).$$

At this point, we may note that the special case in which

(1) all objects are identical with size (l, w), and
(2) L/l and W/w are both integers (with values M and N respectively

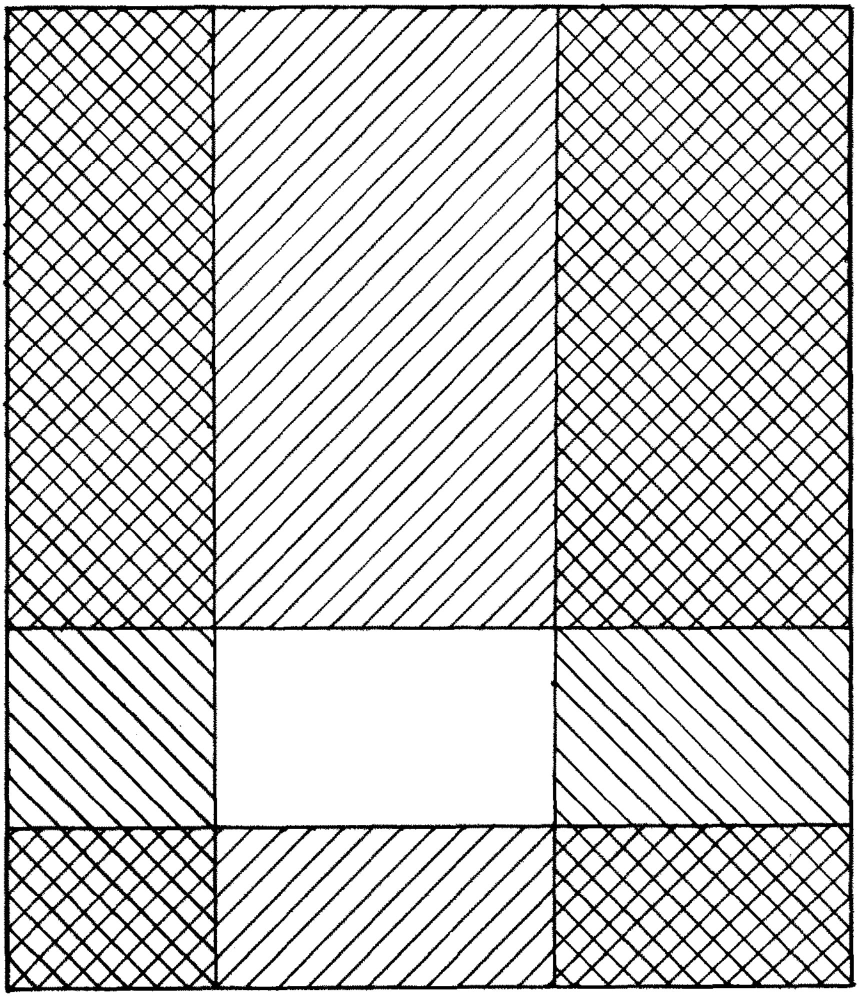

Fig. 1. MERs produced from the placement of a single object in the rectangular area.

is *homomorphic* to the case of 1-by-1 objects placed in an M-by-N area. The quantity R is also bounded: $R_{min} = 1$ (except in certain cases for which $R = 0$ at $B = B_{max}$), and $R_{max} = B_{max}/2$. The maximum of $R(B)$ will be seen to occur for the perfectly antipacked case, the checkerboard pattern. The resulting maximal and minimal envelopes, along with several other representative ones showing the typical behavior of $R(B)$, are shown in Figure 2 for the case $L = W = 4$.

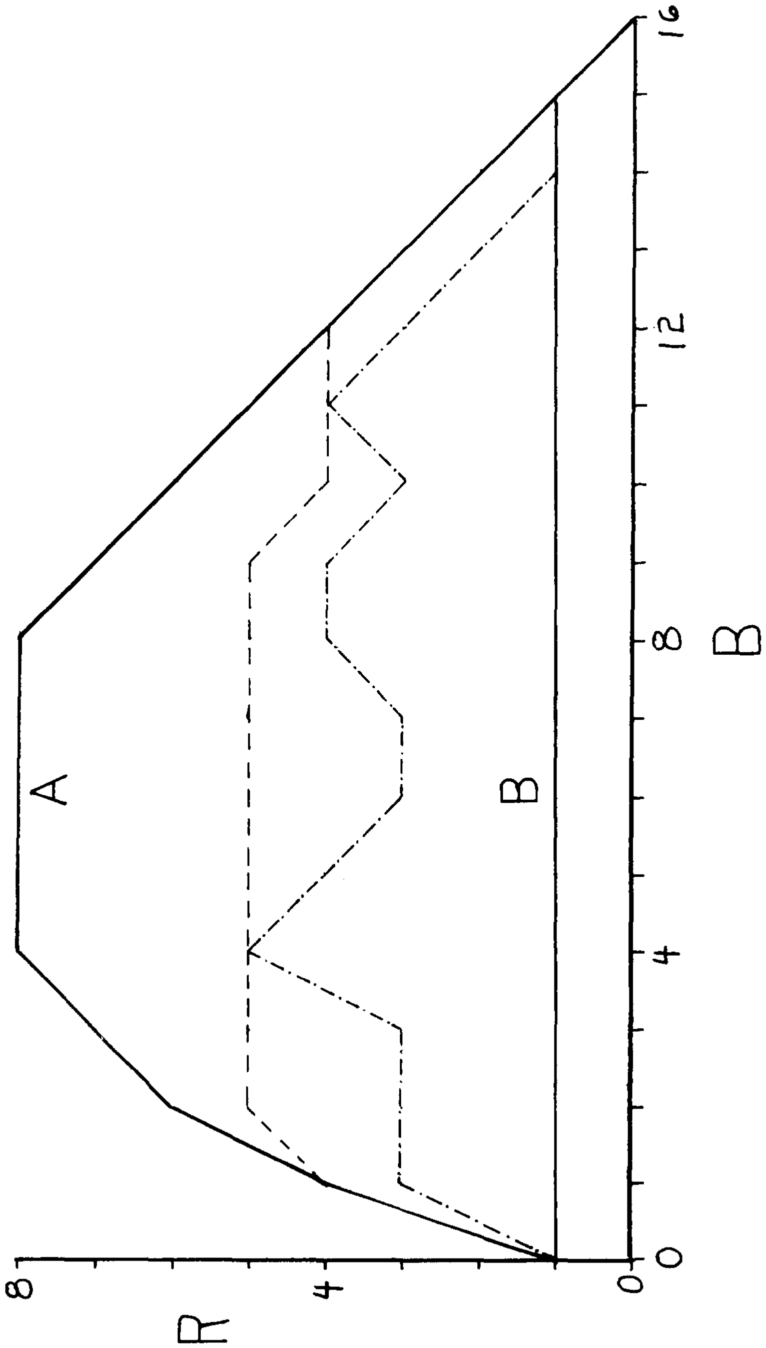

Fig. 2. The maximal (curve A), minimal (curve B), and several intermediate cases (unlabeled curves) of the distribution of the number of MERs produced (R) versus the number of objects placed (B) for 1-by-1 objects placed in a square area whose sides are of length 4.

There are two things worth noticing:

(1) following the *maximal* envelope will always generate the equivalent of the *checkerboard* pattern, and

(2) following the *minimal* envelope will always reduce the L-by-W area to a smaller (rectangular) area L' by W', allowing simple recursive solutions.

2. THE HEURISTIC METHOD

The study of many cases suggested optimizing a certain integral.

PROPOSITION 1. *Optimal packing (antipacking) is obtained when*

$$\int_0^B R(B')\,dB'$$

is a minimum (maximum).

Justification of the Proposition. These facts can be seen by letting $B = B_{\max}$ and examining the minimal and maximal envelopes. For the discrete case, we cannot consider $B_{\max}$ to be large, and by the trapezoidal rule,

$$\int_0^B R(B')\,dB' = \frac{1}{2}\sum_{B'=0}^{B-1}[R(B') + R(B'+1)],$$

or, using the definition of $R(0)$,

$$\int_0^B R(B')\,dB' = \sum_{B'=0}^{B} R(B') - \tfrac{1}{2}[R(B) - 1]$$

We shall take as the heuristic rule that

$$\Delta R = R(B) - R(B-1)$$

will be minimized at each step of the procedure. Note that this is *not* guaranteed to produce optimal packing, but it will make the best choice at each stage and should do fairly well overall. Later, we can recursively examine other cases and choose the one which is best. Figure 3 shows the decision tree for the case of identical 1-by-1 objects with $L = 3$ and $W = 2$. Here the choices have been reduced by symmetry considerations.

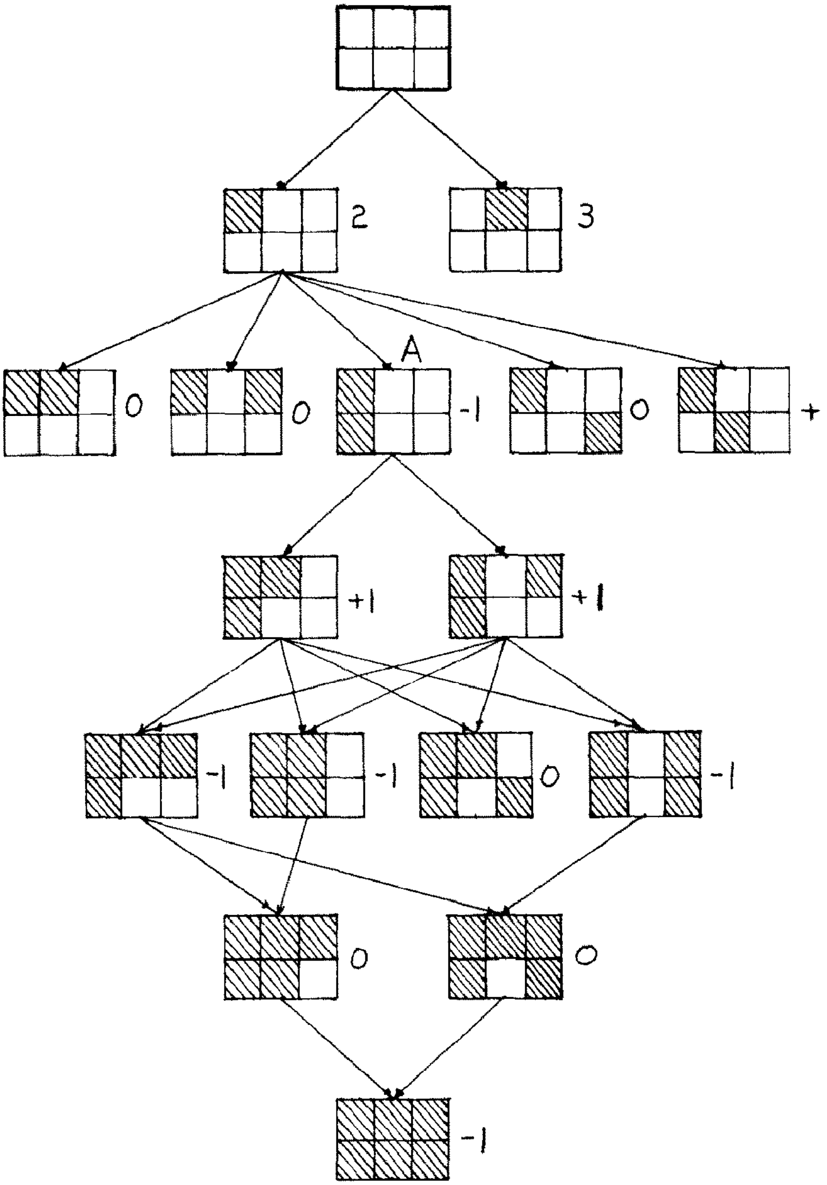

Fig. 3. The decision tree used in packing 1-by-1 objects into a 2-by-3 area. ΔR values are shown to the right of each node (where appropriate). Only nodes with minimum value(s) of ΔR are expanded.

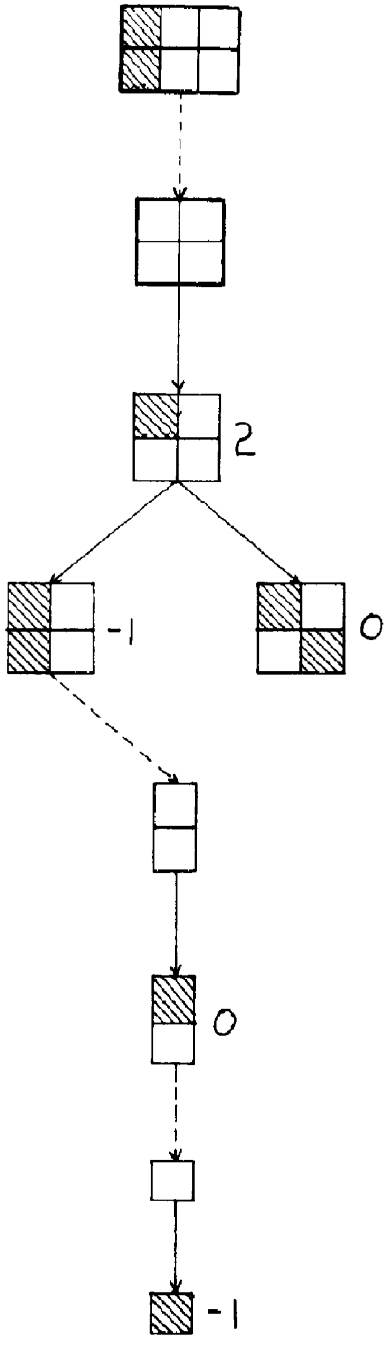

Fig. 4. The tree which results from node A in Figure 3 when recursive redefinition of the area is allowed. The dashed lines indicate such a reduction.

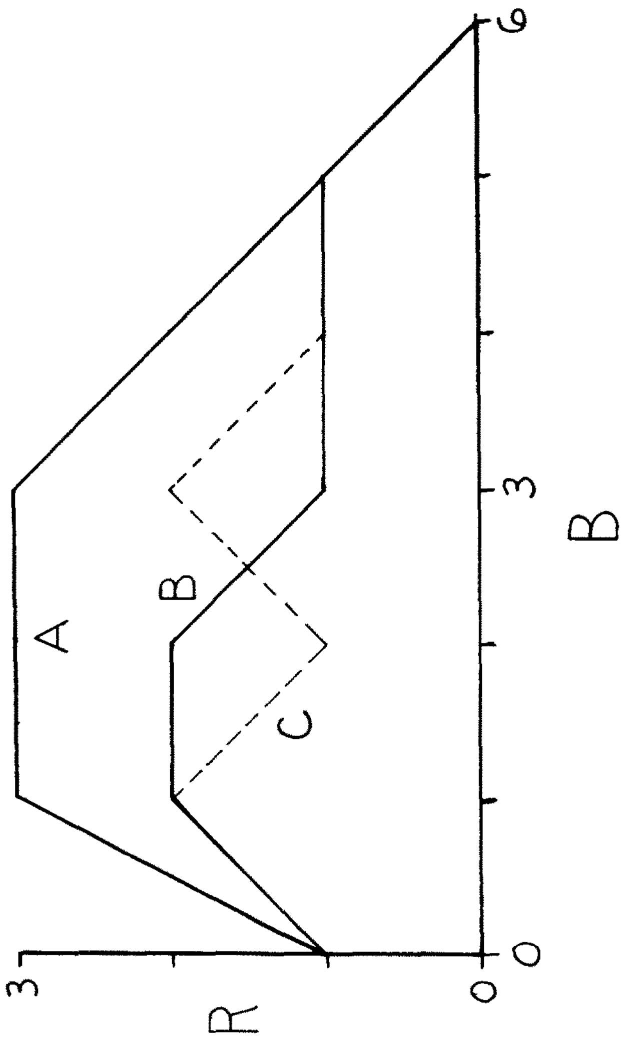

Fig. 5. The distribution $R(B)$ versus B for the case shown in Figures 3 and 4.

PROPOSITION 2. *Once a border is completely filled, it may be brought inward to the edge of the most extended MER, thereby reducing the dimensions of the original area.*

Justification. For example, we can reduce node A of Figure 3 to an area that is 2-by-2 (Figure 4).

In Figure 5, we show the dependence of R upon B for this case. Curve A is the maximal envelope, and curves B and C are two minimal envelopes. The total integrals involved are 12.5 for Curve A and 7.5 for both B and C.

3. ANALYSIS OF THE ALGORITHM

The algorithm itself is very simple. At each step we maintain two lists: one of the MERs sorted in order of increasing area, and the other of the pieces not yet placed, in order of decreasing area. For efficiency, these lists should be linked. We then merely place each object in the smallest MER able to contain it, with the restriction that the number of MERs remaining after the placement is a minimum. After the placement of n objects there will *never* be more than $B_{max}/2$ MERs (in the worst case), and normally quite a few less than this estimate, since it actually applies to the maximal envelope. Thus, on the average, we need only $B_{max}/4$ comparisons for each placement, making this an order-N process, since $B_{max} \approx N$. Therefore, the placement of N objects is of order N^2. These estimates ignore any recursive searches for optimal solutions, which would add another order of N; however, in many cases such searches are unnecessary. For optimal placement of objects, the algorithm is shown to be proportional to the perimeter of the region.

4. DISCUSSION OF THE RESULTS

We have compared the results of this algorithm with the pallet-loading data found in Steudel's paper. Although our problem is not the same as his, we wanted to evaluate our algorithm, and his data seemed to be a worthwhile starting point. However, since we do not allow objects to be rotated, we must assume that some fraction of the objects are in each (i.e., horizontal and vertical) orientation. For a 40-by-48 in. pallet, using 10 objects in the 9-by-6 in. orientation and 24 in the 6-by-9 in orientation, our algorithm places them all and produces an almost identical pattern to that found in Steudel (see Figure 6) with the same space-utilization efficiency (99.6% of available space used). When we tried to place nearly square objects (7.5 by 8.5 in.) using his proportions (14 as 7.5 by 8.5 in. and 15 as 8.5 by 7.5 in.), our algorithm was unable to reproduce his pattern, and in fact, fell far short of his efficiency (96.3%, 29 objects).

A

B

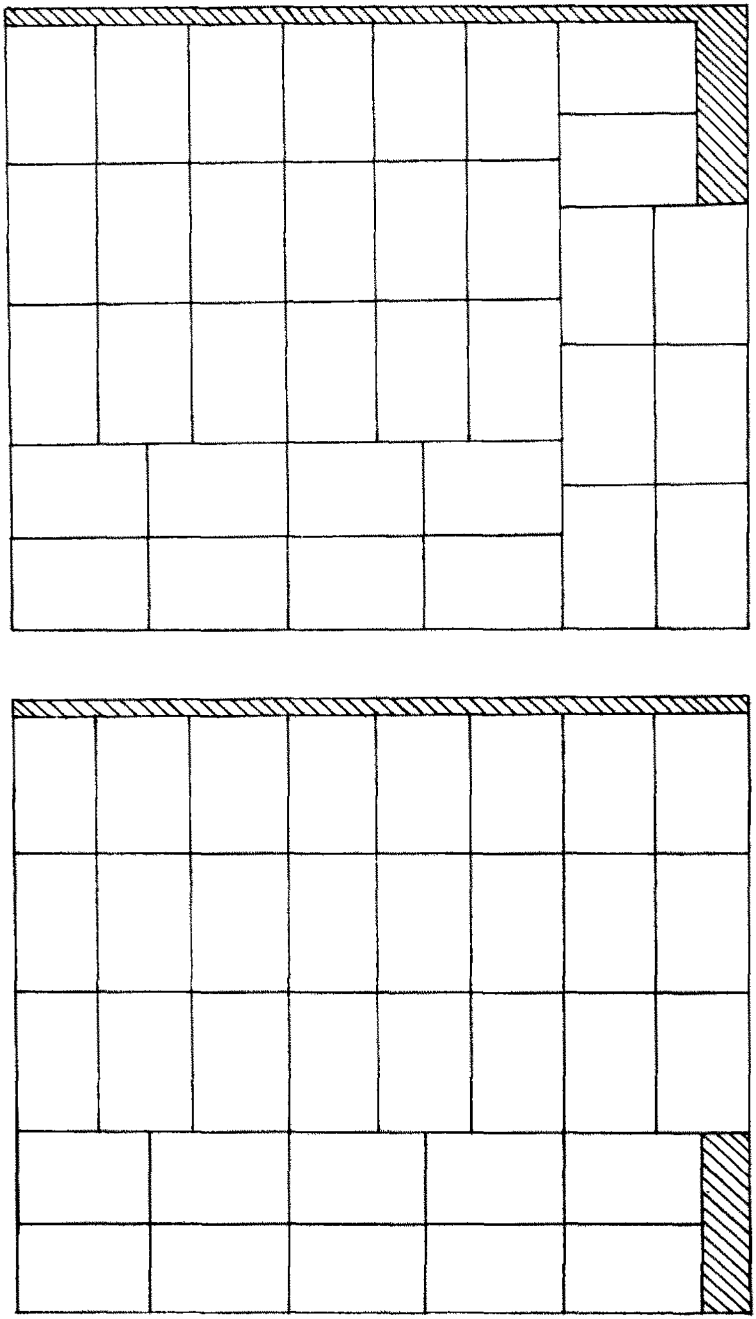

Fig. 6. A comparison of our pallet-loading pattern (A) with that obtained by Steudel (B) for a 40-by-48-in. pallet and 6-by-9-in. objects.

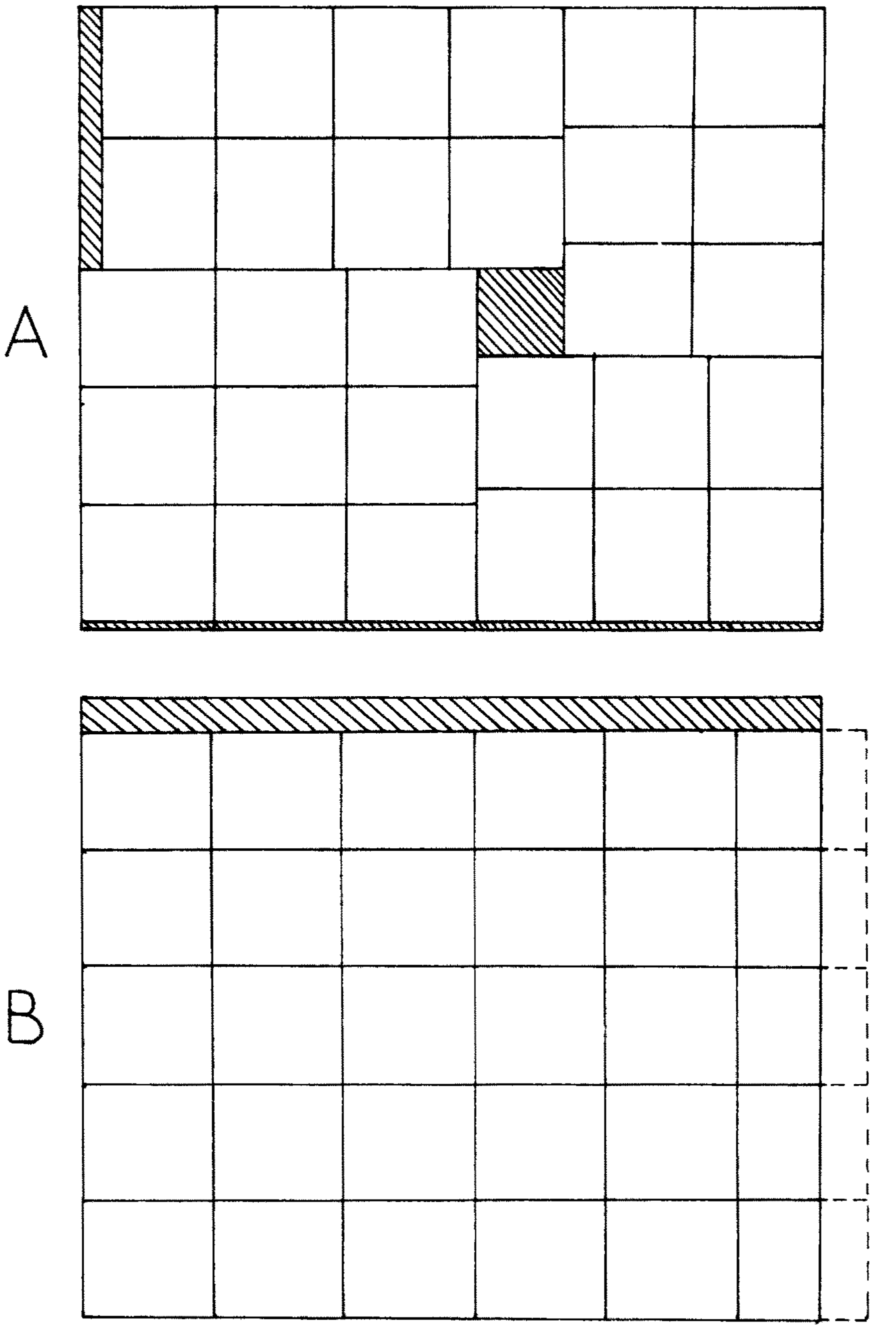

Fig. 7. A comparison of our pallet-loading pattern (A) with that obtained by Steudel (B) for a 40-by-48-in. pallet and 7.5-by-8.5-in. objects. The dashed portion extends over the edge of the pallet (see the text for an explanation).

However, he notes that the Navy loading patterns (obtained from Haynes [3]) allow a slightly oversized load with the overhang not to exceed half of the object's dimension in that direction. For uniformity, we adopt the convention that the overhang will not exceed 3 in. in either direction. On this basis, we can place 30 objects (efficiency 99.6%); the pattern is compared with his in Figure 7. We also reproduce the other pattern he has shown (12 items of length 15.5 in., width 9.5 in.; 92.0% efficiency) in a single pass, where his algorithm required a recursive solution.

5. CONCLUDING REMARKS

The method we have suggested, based on the MER concept, has been examined for several cases and has been found to work well and to be reasonably efficient. For the pallet-loading problem, it produces results which were on the whole as good as those of Steudel. Empirical comparisons of our results indicate that our algorithm is very simple and efficient.

APPENDIX

There exists another description of the free space surrounding an object, which might be called *minimal empty rectangles* (mERs). These are defined by the boundaries of the area and the extensions of the sides of the objects. For example, in Figure 1, we would have 8 mERs. The advantage of these is that the sum of the areas of the objects placed plus the sum of the areas of the mERs produced *always* equals L times W, the area of the original rectangle. However, they possess two disadvantages that make them less desirable than MERs for use in object placement. These are:

(1) they are, in general, more numerous than MERs (e.g., 8 mERs versus 4 MERs in Figure 1), and, more importantly,

(2) they must be concatenated when trying to place objects.

We wish to thank Professor Jayakumar for supplying a copy of his paper. Laurent was a graduate student in the Department of Physics and Astronomy, Louisiana State University, when this work was done.

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Received May 1981; revised 7 November 1981