Sparsification of Influence Networks

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ABSTRACT

We present SPINE, an efficient algorithm for finding the “backbone” of an influence network. Given a social graph and a log of past propagations, we build an instance of the independent-cascade model that describes the propagations. We aim at reducing the complexity of that model, while preserving most of its accuracy in describing the data.

We show that the problem is inapproximable and we present an optimal, dynamic-programming algorithm, whose search space, albeit exponential, is typically much smaller than that of the brute force, exhaustive-search approach. Seeking a practical, scalable approach to sparsification, we devise SPINE, a greedy, efficient algorithm with practically little compromise in quality.

We claim that sparsification is a fundamental data-reduction operation with many applications, ranging from visualization to exploratory and descriptive data analysis. As a proof of concept, we use SPINE on real-world datasets, revealing the backbone of their influence-propagation networks. Moreover, we apply SPINE as a pre-processing step for the influence-maximization problem, showing that computations on sparsified models give up little accuracy, but yield significant improvements in terms of scalability.

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1. INTRODUCTION

For many scenarios of network analysis, sparsification is a fundamental data-reduction operation that equips the data analyst with the ability to visualize, explore, digest, and interpret more easily the available data. In addition, sparsification helps in reducing the noise in the data and in avoiding over-fitting, thus allowing to build more accurate models.

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In this paper we study a sparsification framework that is appropriate for analyzing information propagation in social networks. We aim at eliminating a large number of links in the network, and preserving only the links that play an important role on how information propagates. The importance of a link is measured by its ability to explain observed propagations. Sparsifying a network with respect to a log of “information actions” can be seen as revealing the backbone of information propagation in the network.

A high-level description of our approach is the following. We are given a social network, that is, a set of “friendship” links or “follower-follower” links. Additionally, we assume that we are given a log of actions performed by the nodes of the network. Such actions may include posting information “memes”, joining an online community or an interest group, buying a paid subscription in an on-line service, and so on. We assume that those actions have propagated in the network via the independent cascade model [18]. The maximum likelihood parameters of this model can be found for instance by using the EM algorithm of Saito et al. [28].

Given the parameters we formulate the sparsification problem, which is the focus of this paper: we ask to preserve the $k$ most important links in the model, i.e., the set of $k$ links that maximize the likelihood of the observed data. Here $k$ might be an input parameter specified by the data analyst, or alternatively $k$ might be set automatically following common model-selection practice.

Our framework has a number of interesting applications.

Propagation characterization: Sparsifying separately different information topics can help us answer questions such as: “What distinguishes the way important news propagate, from the way funny “memes” do?” or “Is there any structural difference between the backbone of actual news and that of false rumors?”

Feed ranking: As users in social networks receive a continuous feed of information, ranking the most interesting feeds is becoming an important problem [15]. Sparsification provides a useful feature in this ranking problem by highlighting the most important links.

Viral marketing: Finding the set of users to target in order to maximize the spread of influence is a problem that has received a lot of attention, yet there are still serious computational challenges [3, 4]. As we show in Section 7, sparsification yields significant improvement for this problem, in terms of efficiency and scalability, while sacrificing little in terms of accuracy.

Even though there has been a lot of work recently on studying information propagation, mostly devoted either to
empirical analysis of real-world propagations [1, 14, 19, 22], or to devise methods for influence maximization [6, 18, 27], not much effort has been devoted to develop techniques to mine large logs of propagation traces. On the other hand, there is extensive literature on the problem of network simplification, but the scenarios assumed are different than the problem we study in this paper. We review this literature in the next section.

Our contributions are summarized as follows.

- Given a social network and a log of actions, we study the problem of pruning the network to a prefixed extent while maximizing the likelihood of generating the propagation traces in the log (Sections 3 and 4).
- We show that our problem is \( \text{NP} \)-hard to approximate within any multiplicative factor (Section 4).
- We show that sparsification can be decomposed into a number of subproblems equal to the number of the nodes in the network. We then present an exponential, but optimal, dynamic programming algorithm, whose search space is typically much smaller than the brute force one, but still impracticable for graphs having nodes with a large in-degree (Section 5.1).
- We devise \text{Spine} (Sparsification of Influence networks), a greedy algorithm that achieves efficiency with practically little compromise in quality. \text{Spine} is structured in two phases. During the first phase it selects a set of arcs \( D_0 \) that yields a finite log-likelihood; during the second phase, it greedily seeks a solution of maximum log-likelihood. The solution returned by \text{Spine} is guaranteed to be “close” to the optimal among the subnetworks that contain arcs \( D_0 \) (Section 5.2).
- We show that \text{Spine} identifies efficiently sparse networks of high or even optimal likelihood (Section 6).
- We apply \text{Spine} as a pre-processing step for the problem of influence maximization, showing that computations on the sparse network give up little accuracy, but yield significant improvements in terms of efficiency and scalability (Section 7).

2. RELATED WORK

The study of information diffusion in social networks has a long history in social sciences. Early work studied the adoption of medical and agricultural innovations [5, 32]. Later, marketing researchers investigated the “word-of-mouth” diffusion process for viral marketing [2, 9, 17, 23].

From a computational perspective, a basic problem in viral marketing is that of influence maximization: given a social network, find \( k \) nodes to target in order to maximize the spread of influence. The first algorithmic treatment of the problem was provided by Domingos and Richardson [6, 27], who modeled the diffusion process in terms of Markov random fields, and proposed heuristic solutions to the problem. Subsequently, Kempe et al. [18] studied the influence-maximization problem for a different family of influence models. Part of their contribution was to provide approximation algorithms for the independent cascade model, which we also adopt in this paper. Recent work [3, 4, 21] improves the efficiency of influence maximization.

Conceptually, our work can be collocated with works on network simplification, the goal of which is to identify subnetworks that preserve properties of a given network. Toivo-

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$t$ and node $v$ follows $u$, then $u$ makes a single attempt to influence $v$, and succeeds with probability $p(u,v)$. The probability of success is independent of other nodes that may attempt to influence $v$. If it is successful, and $v$ has not previously performed $\alpha$, then $v$ performs $\alpha$ in time step $(t+1)$. Note that, according to this model, several nodes may attempt to influence $v$, but $v$ may perform $\alpha$ at most once.

Additionally, we assume that propagations are initiated by a special node $\Omega \in V$ (Figure 1(a)) that has the following two properties: (i) it performs $\alpha$ before any other node, and (ii) is followed by all other nodes in $V$. Node $\Omega$ models sources of influence that are external to the network, and thus $p(\Omega, v)$ is the probability that a propagation starts from node $v$.

The independent cascade model generates sequences of activations, or traces, as we call them. Traces of different actions are generated independently. Note that, given the trace of action $\alpha$, it may not be possible to tell which node influenced a particular node $v$ to perform $\alpha$, since more than one node may have attempted to do so. In the example of Figure 1(b), for instance, one of nodes: $u_1$ or $u_2$, succeeded to influence $v$. We say that $u_1$ and $u_2$ possibly influenced $v$. On the other hand, neither $u_3$ nor $u_4$ could have influenced $v$, because then, according to the independent cascade model, $v$ would have performed $\alpha$ at time $t$.

Every trace generated by the independent cascade model is associated with a likelihood value. Consider the trace of action $\alpha$. Let $F_{\alpha}^+(v)$ be the set of nodes that possibly influenced $v$, and $F_{\alpha}^-(v)$ the set of nodes that definitely failed to influence $v$. Then, the likelihood $L_{\alpha}(G)$ of the trace can be written as

$$L_{\alpha}(G) = \prod_{v \in V} P_{\alpha}^+(v) \cdot P_{\alpha}^-(v),$$

where

$$P_{\alpha}^+(v) = \begin{cases} 1 & \text{if } F_{\alpha}^+(v) = \emptyset, \\ 1 - \prod_{u \in F_{\alpha}^+(v)} (1 - p(u,v)) & \text{otherwise}, \end{cases}$$

expresses the likelihood that at least one of the nodes in $F_{\alpha}^+(v)$ succeed to influence $v$, and

$$P_{\alpha}^-(v) = \prod_{u \in F_{\alpha}^-(v)} (1 - p(u,v)).$$

the likelihood that all nodes in $F_{\alpha}^-(v)$ fail. For the rest of the paper, following common practice, we opt to work with log-likelihood, due to its better numerical behavior

$$\log L_{\alpha}(G) = \sum_{v \in V} (\log P_{\alpha}^+(v) + \log P_{\alpha}^-(v)).$$

We now describe how to estimate the probabilities $p(u,v)$ of the independent cascade model from a set of traces. Consider a set of actions $A$. For each action $\alpha \in A$ we observe its propagation trace, and assume that all traces are generated by the same model. Following [28], the probability values $p(u,v)$ that maximize the total log-likelihood

$$\log L(G) = \sum_{\alpha \in A} \log L_{\alpha}(G)$$

can be computed using the following iterative formula

$$p^{(k+1)}(u,v) = \frac{p^{(k)}(u,v)}{\mid A_{\mid u} \mid + \mid A_{\mid u} \mid} \sum_{\alpha \in A_{\mid u}} \frac{1}{P_{\alpha}^+(v)}.$$
4. SPARSIFICATION

Given an influence model learned (as in Section 3) from a social network \( G = (V, D) \) and propagation traces for a set of actions \( A \), our goal is to identify a “backbone” of arcs that are most important for the propagation of actions \( A \). Specifically, we aim to identify a sparse subnetwork \( G_s \) of \( G \), that consists of the \( k \) arcs that are most likely to have generated the observed traces of actions \( A \).

Formally, we define a network \( G_s = (V, D_s) \) to be a sparse subnetwork of \( G = (V, D) \) if its arcs \( D_s \) are a subset of \( D \), \( D_s \subseteq D \), and the probabilities \( p_s(u,v) \) are equal with the corresponding probabilities \( p(u,v) \) of the network \( G \), that is, \( p_s(u,v) = p(u,v) \), for all \( (u,v) \) in \( D_s \). The problem of sparsifying network \( G \) is defined as follows.

Problem 1 (Sparsification). Given a network \( G = (V, D) \) with probabilities \( p(u,v) \) on the arcs, a set \( A \) of action traces, and an integer \( k \), find a sparse subnetwork \( G_s = (V, D_s) \) of \( G \) of size \( |D_s| = k \), so that the log-likelihood function \( \log L(G_s) \) is maximized.

Notice that Problem 1 is not solved by selecting the \( k \) arcs \( (u,v) \) in \( D \) with the largest probability values \( p(u,v) \). As a counter-example, consider the network \( G = (V, D) \) and traces of actions \( A = \{\alpha_1, \alpha_2, \alpha_3\} \), shown in Figure 3. Based on the three traces, we obtain the maximum likelihood estimates \( p(u_1,v) = 1.0 \) and \( p(u_2,v) = 0.5 \). Then, for \( k = 3 \), the best sparse model \( G_s = (V, D_s) \) is the one with

\[ D_s = \{(\Omega, u_1), (\Omega, u_2), (u_2, v)\} = D \setminus \{(u_1, v)\} \]

even though \( p(u_2,v) < p(u_1,v) \). To see why, notice that, for example, the alternative option of

\[ D_s = \{(\Omega, u_1), (\Omega, u_2), (u_1, v)\} = D \setminus \{(u_2, v)\} \]

leads to zero likelihood \( \log L(G_s) = -\infty \). This is because the trace of action \( \alpha_2 \) is impossible without arc \( (u_2,v) \). On the other hand, all three traces are possible in the absence of the arc \( (u_1,v) \).

Figure 3: Sparsification is not solved by selecting the \( k \) arcs with largest probability value.

Complexity. By the definition of the independent cascade model, for a sparse network \( G_s = (V, D_s) \) to have finite log-likelihood, \( \log L(G_s) > -\infty \), it is necessary that the traces of all actions \( A \) are possible for its set of arcs \( D_s \). This means that if node \( v \) performs an action \( \alpha \) in \( A \), then \( D_s \) must include an arc from at least one of the nodes \( F^\alpha_s \) that possibly influenced \( v \). Our argument is formally stated below.

Lemma 1. Deciding whether Problem 1 has finite solution is \( \text{NP} \)-hard.

Proof. We obtain a reduction from the Hitting Set problem. We first remind the Hitting Set problem: Given a collection of sets \( S = \{S_1, \ldots, S_m\} \) over a universe of \( n \) elements \( U = \{1, \ldots, n\} \) (i.e., \( S_j \subseteq U \)), a hitting set for \( S \) is a set \( H \subseteq U \) that intersects all sets in \( S \), that is, \( H \cap S_j \neq \emptyset \) for all \( j = 1, \ldots, m \). Given a set collection \( S \) and an integer \( k \), it is \( \text{NP} \)-hard to decide whether there is a hitting set \( H \) for \( S \) that has size at most \( |H| \leq k \).

Now, consider an instance of the Hitting Set problem, i.e. consider a collection \( S = \{S_1, \ldots, S_m\} \) and an integer \( k \). We create an instance of our problem as follows. Our graph \( G = (V, D) \) has \( n+1 \) nodes, namely, \( V = U \cup \{n+1\} = \{1, \ldots, n, n+1\} \). The nodes from 0 to \( n \) have no ancestors, while the node \( n+1 \) has all nodes from 0 to \( n \) as ancestors. Thus \( D = \{(i, n+1) \mid i = 1, \ldots, n\} \). For all the edges in \( D \) we set \( p(i,n+1) = 1 \). Next, for each set \( S_j \in S \) we consider an action \( \alpha \) that was performed by nodes \( S_j \cup \{n+1\} \). First consider a hitting set \( H \) of \( S \), with \( |H| \leq k \). Take the sparsiﬁed graph \( G_s = (V, D_s) \) with \( D_s = \{(i, n+1) \mid i \in H\} \). It is \( |D_s| \leq k \). Furthermore for each action \( \alpha \in A \) the set \( D_s \) contains at least one edge from a parent of node \( n+1 \) that is also inﬂuenced by \( \alpha \). Given that all arcs have probability 1 it follows that the probability \( P^\alpha_s(n+1) \) is equal to 1, and consecutively the total log-likelihood is ﬁnite.

Conversely, for any sparsiﬁed graph with at most \( k \) edges and ﬁnite log-likelihood it should be the case that for each action \( \alpha \in A \) the set \( D_s \) contains at least one edge from a parent of node \( n+1 \) that also performed \( \alpha \). Thus the set \( H = \{i \mid (i, n+1) \in D_s\} \) is a hitting set for \( S \).

Lemma 1 leads to the following hardness results.

Theorem 1. Problem 1 is \( \text{NP} \)-hard.

Theorem 2. Approximating Problem 1 up to any multiplicative factor is \( \text{NP} \)-hard.

The latter Theorem follows from the simple observation that obtaining a multiplicative-factor approximation is at least as diﬃcult as obtaining a ﬁnite solution. Theorem 1 is a special case of Theorem 2.

5. ALGORITHMS

5.1 An optimal algorithm

A brute-force approach to acquire an optimal solution to Problem 1 is to enumerate all possible subsets of arcs \( D_s \subseteq D \) of size \( k \), and select the network \( G_s = (V, D_s) \) with maximum log-likelihood \( \log L(G_s) \). This approach is exponential in the size of \( D \) and obviously does not scale.

Instead, we describe an optimal, dynamic programming algorithm, OPTIMALSPARSE, that makes relatively limited use of exhaustive enumeration. The algorithm is based on the observation that an optimal solution can be obtained by dividing Problem 1 into \(|V|\) sub-problems, where \( V \) is the set of nodes of \( G \), and combining their solutions. To see this, let us re-write log-likelihood as

\[
\log L(G) = \sum_{\alpha \in \mathcal{A}} \log L_{\alpha}(G) = \\
= \sum_{\alpha \in \mathcal{A}} \sum_{v \in V} (\log P^\alpha_s(v) + \log P^\alpha_s(v)) \\
= \sum_{v \in V} \sum_{\alpha \in \mathcal{A}} (\log P^\alpha_s(v) + \log P^\alpha_s(v)).
\]

Observe that the inner sum of the above formula corresponds to a single node \( v \). Also, recall that the probabilities \( P^\alpha_s(v) \) and \( P^\alpha_s(v) \) are “local” to node \( v \), meaning that they depend
only on influence probabilities on arcs from the parent nodes of \( v \) in \( G \).
Denote the set of these arcs by \( D_v \), and observe that we can compute \( P^*_o(v) \) and \( P^*_v(v) \) for an arbitrary subset \( V_o \subseteq D_v \) simply by omitting probabilities on edges not present in \( V_o \). This way we can define

\[
\lambda(X_o) = \sum_{a \in A} \left( \log P^*_o(v \mid X_o) + \log P^*_v(v \mid X_o) \right),
\]

(7)

for \( X_o \subseteq D_v \). The log-likelihood of the full model can thus be written as \( \log L(G) = \sum_v \lambda(D_v) \). Therefore we consider the following sub-problem for each node \( v \in V \).

**Problem 2.** Given an integer \( b \), identify a subset of arcs \( D^b_v \subseteq D_v \) of size \( b \), such that \( \lambda(D^b_v) \) is maximized (Algorithm 1, Lines 7-8). By construction, we describe

finding experimentally that the quality of the results obtained

is comparable to the that of the brute-force approach, which increases exponentially with the size of \( D_v \), but is typically much smaller than that of the brute-force approach, which increases exponentially with the size of \( D_v \).

Algorithm \textsc{OptimalSparse} uses dynamic programming to compute an optimal solution \( D_v \) to Problem 1 from optimal solutions of Problem 2. Specifically, let \( V = \{v_1, v_2, \ldots\} \) be an enumeration of the nodes. For each node \( v_i \), consider the solution \( D^b_v \) to Problem 2 for all integers \( b \in [1, k] \) (Algorithm 1, Lines 1-3). \textsc{OptimalSparse} then proceeds sequentially over nodes \( V \) (Algorithm 1, Line 5). At the end of the \( i \)-th step, it has processed nodes \( v_1, v_2, \ldots, v_i \), and, for each integer \( m \in [1, k] \) (Algorithm 1, Line 6) it identifies one number \( b \) for each node \( v \in \{v_1, v_2, \ldots, v_i\} \) such that \( b_1 + b_2 + \ldots + b_i = m \) and the sum

\[
\Lambda(i, m) = \lambda(D^b_v) + \lambda(D^{b_2}_v) + \ldots + \lambda(D^{b_i}_v)
\]

is maximized (Algorithm 1, Lines 7-8). By construction, \( \Lambda(|V|, k) \) contains the maximum log-likelihood value for Problem 1. The optimal solution \( D_v \) is computed with standard back-tracking (not described in Algorithm 1, in interest of brevity).

**Algorithm 1 OptimalSparse**

1. for \( i = 1 \) to \( |V| \) do
2. for \( b = 1 \) to \( k \) do
3. compute optimal \( D^b_v \) (Problem 2)
4. array \( \Lambda \). initialize \( \Lambda(0, m) = 0; m = 1 \) to \( k \)
5. for \( i = 1 \) to \( |V| \) do
6. for \( m = 1 \) to \( k \) do
7. \( b_i := \arg \max \{\lambda(D^b_v) + \lambda(i - 1, m - b_i)\} \)
8. \( \Lambda(i, m) := \lambda(D^b_v) + \lambda(i - 1, m - b_i) \)
9. return \( \Lambda(|V|, k) \)

### 5.2 A greedy algorithm: SPINE

\textsc{OptimalSparse} follows a more sophisticated approach than brute-force, but is still prohibitively expensive for graphs that have nodes with large in-degree. In this section, we describe \textsc{Spine}, a greedy algorithm for Problem 1. Although Problem 1 is inapproximable in the general case, we find experimentally that the quality of the results obtained by \textsc{Spine} is comparable to that of \textsc{OptimalSparse}.

\textsc{Spine} produces a solution \( D_v \) to Problem 1 in \( k \) steps, adding to \( D_v \) one arc at each step. Those \( k \) steps are divided in two phases: during the first phase, \textsc{Spine} aims to identify a solution \( D_0 \) of finite log-likelihood; during the second phase, it greedily seeks a solution of maximum log-likelihood. This two-phase approach is inspired by the observation that Problem 1 is at least as difficult as identifying a solution of finite log-likelihood.

**First phase.** Following the discussion on complexity of Problem 1 (Section 4), we look for a solution with finite log-likelihood by solving an instance of the hitting set — that is, for each node \( v \in V \) we seek for a hitting set of collection

\[
C(v) = \{D^b_v \mid b \neq \emptyset; \ a \in A\}.
\]

As \textsc{Hitting Set} is \textsc{NP}-hard, we employ the greedy approximation algorithm described in [16] (Algorithm 2, Lines 1-8). According to that algorithm, arcs \( (u, v) \) are ordered by the number \( n(u, v) \) of actions for which \( u \) possibly influenced \( v \)

\[
n(u, v) = |\{D^b_v \in C(v) \mid (u, v) \in D^b_v\}|.
\]

At each step, the arc \( (u, v) \) with the maximum number \( n(u, v) \) is selected (Algorithm 2, Lines 6-7) and all sets \( D^b_v \) that contain \( (u, v) \) are ignored for the rest of this process (Algorithm 2, Line 8). The first phase ends when either the limit of \( k \) arcs is reached, or \( \lambda(D^k_v) \) for \( b \leq k \) selected arcs becomes finite for the first time.

**Second phase.** Let \( D_0 \) be the set of arcs selected by the end of the first phase, and let \( G_0 = (V, D_0) \) be the associated sparse network. If \( |D_0| < k \), then we still need to select \( k - |D_0| \) more arcs. One viable approach to select the remaining \( k - |D_0| \) arcs is to continue in a fashion similar to the \textsc{OptimalSparse} algorithm: divide the problem into \( |V| \) subproblems, one for each node in the graph, solve optimally each subproblem for all values of \( b = 1, \ldots, k - |D_0| \), and then combine the solutions of the subproblems with dynamic programming. However, due to the exhaustive search and the dynamic programming, such an approach would not be scalable. To obtain a scalable solution we propose to speed up the computation by replacing both steps — exhaustive search and the dynamic programming — with a greedy process. We choose the remaining \( k - |D_0| \) arcs by selecting greedily at each step the arc that offers the largest increase in log-likelihood (Algorithm 2, Lines 9-15).

For a detailed description, let \( D_v \) and \( \lambda \) be defined as above, and consider a table \( X \), where \( X_v(i) \) is the subset of \( D_v \) that the greedy algorithm would add the \( i \)-th step when maximizing \( \lambda \). More formally, we have

\[
X_v(i) = \begin{cases} 
\emptyset & \text{if } i = 0, \\
X_v(i - 1) \cup e_v(i) & \text{otherwise,}
\end{cases}
\]

where \( e_v(i) = \arg \max_{e \in U_v} \{\lambda(X_v(i - 1) \cup e)\} \), and \( U_v = D_v \setminus X_v(i - 1) \). Also, let \( H_v(i) \) denote the marginal gain for \( \lambda \) when the \( i \)-th edge is added, that is, let

\[
H_v(i) = \begin{cases} 
\lambda(X_v(i)) - \lambda(X_v(i - 1)) & \text{if } i = 0, \\
0 & \text{otherwise.}
\end{cases}
\]

\textsc{Spine} maintains the \( X_v \) and \( H_v \) structures for every \( v \), and at every step chooses the \( v^* \) and \( i^* \) that maximize \( H_v(i^*) \) (line 13). The edge \( e_v(i^*) \) will be added to the solution, and we compute \( \text{S}_{v^*}(i^* + 1) \) and \( \text{H}_{v^*}(i^* + 1) \). Computing \( X_v(i + 1) \) given \( X_v(i) \) is an \( \mathcal{O}(|D_v|) \) operation, while inserting and extracting from \( Q \) are \( \mathcal{O}(\log |V|) \). The worst-case complexity of the 2nd phase of \textsc{Spine} is thus \( \mathcal{O}(k(\max(|D_v|) + \log |V|)) \).

We have the following approximation guarantee.
Lemma 2. Let $D_{opt}$ be a superset of $D_0$ that contains $k$ arcs and that induces a subgraph $G_{opt} = (V, D_{opt})$ of $G$ with maximum log-likelihood. Also, let $D_0$ be the set of arcs returned by SPINE and let $G_0 = (V, D_0)$ be the induced subgraph. That is, $D_0$ is also a superset of $D_0$ and it has $k$ arcs. Then, provided that $\log L(G_0) > -\infty$, we have

$$\log L(G_{opt}) \geq \frac{1}{e} \log L(G_0) + (1 - \frac{1}{e}) \log L(D_{opt}).$$ (9)

Proof. We use a well-known theorem by Nemhauser et al. [25] on maximizing submodular set functions. A set function $f$ is submodular, when for any $S \subseteq T$ we have $f(S \cup u) - f(S) \geq f(T \cup u) - f(T)$. The theorem in [25] states that for non-negative set functions $f$, with $f(\emptyset) = 0$, we have $f(X_k^0) \geq (1 - 1/e)f(X_k^0)$, where $X_k^0$ is the optimal k-sized solution, and $X_k^0 \cup S$ a solution likewise of size $k$ that is constructed by starting from $\emptyset$, and at each step adding the item that maximizes the increase in the value of $f$. Observe that this greedy strategy is also used by SPINE when adding edges to the sparse model.

A direct application of this result is impossible, as log-likelihood is negative, and not equal to zero for an empty solution. However, we consider the following modification: Denote by $LL(G)$ the log-likelihood $log L(G)$, and let

$$g(S) = LL(G_0 \cup G_S) - LL(G_0),$$

where $S \subset D$ is a set of edges, $G_S$ is the graph corresponding to the edges in the set $S$. Furthermore, observe that $S$ can be seen as the set of edges added by SPINE in the 2nd phase. Clearly $g(S)$ is non-negative, and $g(\emptyset) = 0$. We can also show the following lemma, whose proof can be found in the extended version of this paper.

Lemma 3. Function $g(S)$ is submodular.

Now we apply the result of [25] directly, and have

$$g(S^A) \geq (1 - 1/e)g(S^*)$$ (11)

Expanding $g(S)$, inequality (11) gives

$$LL(G_{sp}) \geq (1 - 1/e)(LL(G_{opt}) - LL(G_0)) + LL(G_0)$$

$$= \frac{1}{e}LL(G_0) + (1 - \frac{1}{e})LL(G_{opt}),$$

where $G_{sp} = G_0 \cup G^A$ is the solution returned by SPINE, and $G_{opt} = G_0 \cup G^*$ is the optimal solution that contains $G_0$ as a subset, which concludes the proof of Lemma 2.

Lemma 2 guarantees that the solution returned by SPINE is close to the optimal among the subnetworks that contain arcs $D_0$. Notice that this result is not a true approximation result for Problem 1. First, Lemma 2 does not associate $log L(G_{sp})$ with log $L(G_a)$, the log-likelihood of the optimal solution, and secondly, it is true under the provision that $log L(G_0) > -\infty$, i.e., that the first phase of SPINE “escapes” infinite log-likelihood before the $k$-th step.

A note on parallelization. In practice we have observed that the total execution time of SPINE is dominated by Phase 2 by at least an order of magnitude. Therefore we only concentrate on optimizing this part of the algorithm. Note that the computationally involved part in the 2nd phase is updating the table $X$ on line 14. In Algorithm 2 we compute entries of $X$ on-demand, i.e., the entry $X_v(i)$ is only computed when we must find out if the $i$-th parent of $v$ should belong to the solution. Alternatively we could compute the entire table $X$ in advance, and then run phase 2 of SPINE with this as the input. This approach has the advantage that the values $X_v(1), X_v(2), \ldots$ can be computed independently of other vertices for each $v \in V$, hence SPINE lends itself to easy parallelization by arbitrarily partitioning $V$.

Setting the value of $k$. The size $k$ of sparse network $G_k$ is given as input to problem 1. A natural question that arises, then, is how to specify $k$ in a principled manner, so as to identify sparse networks that combine small size with high log-likelihood. Following common model-selection practice, one way to set $k$ is via minimization of the Bayesian Information Criterion (BIC).

$$BIC(G_k) = -2\log L(G_k) + k\log(|A|)$$ (12)

It is straightforward to modify our algorithms to use BIC instead of log-likelihood, and return a sparse network $G_k$ that minimizes the BIC. Part of our experiments (Section 6) is devoted to assessing SPINE in terms of BIC.

6. EXPERIMENTS

In this section we present an evaluation of the performance of SPINE on real datasets.

6.1 Experimental framework

Datasets. We test our algorithms on samples extracted from two different datasets. The first data source, referred to as YMEME in the following, is a set of microblogging postings in Yahoo! Meme. Nodes are users, actions correspond to postings (typically photos) that users share through the site, and arcs from a node $u$ to a node $v$ indicate that $v$ follows $u$. The data source contains all posts in this platform from March 2009, obtained by Meme Tracker [20].

The second data source, referred to as MTRACK in the following, is a set of phrases propagated over prominent online news sites in March 2009, obtained by Meme Tracker [20]. Nodes are mostly news portals or news blogs and actions correspond to phrases that are found by the Meme Tracker algorithm to be repeated across several sites. In absence of explicitly declared “follower-followee” relationships between

\[http://meme.yahoo.com/\]
the sites, arcs from a node \( u \) to a node \( v \) indicate that the website \( v \) linked to the website \( u \) during March 2009, and thus \( v \) "follows" \( u \). This dataset is publicly available.\(^2\)

We used a snowball sampling procedure to obtain several subsets from these data sources. In the case of YMEME, we sampled a connected sub-graph of the social network containing the users that participated in the most reposted items. This yields very densely connected subgraphs. In the case of MTRACK, we sampled a set of highly reposted items posted by the most active sites. This yields more loosely connected subgraphs. A summary of the subsets we created is shown in Table 1.

To estimate the probabilities associated with arcs, we use the EM algorithm of [28] (Section 3). In interest of simplicity, here we show results only for delay threshold \( \Delta t = \infty \) and study effects of \( \Delta t \) in an extended version of the paper. As explained in Section 3, \( \Delta t = \infty \) means that when node \( v \) performs action \( \alpha \), then \( F^+_{\alpha}(v) \) contains all nodes followed by \( v \) that performed \( \alpha \) before \( v \), and \( F^-_{\alpha}(v) \) is empty. On the other hand, if \( v \) does not perform action \( \alpha \), then \( F^+_{\alpha}(v) \) is empty, and \( F^-_{\alpha}(v) \) contains all nodes followed by \( v \) that performed \( \alpha \) and thus failed to influence \( v \).

As a product of the maximum-likelihood estimation, it may be that some arcs are associated with zero (0) influence probability. The last column of table 1 reports the number of arcs with positive probability after estimation terminates.

### Table 1: Summary of datasets used.

<table>
<thead>
<tr>
<th>Dataset</th>
<th>Actions</th>
<th>Nodes</th>
<th>Arcs input</th>
<th>Arcs prob. &gt; 0</th>
</tr>
</thead>
<tbody>
<tr>
<td>YMEME-L</td>
<td>26000</td>
<td>10776</td>
<td>427711</td>
<td>342613</td>
</tr>
<tr>
<td>YMEME-M</td>
<td>13000</td>
<td>9525</td>
<td>1154674</td>
<td>378520</td>
</tr>
<tr>
<td>YMEME-S</td>
<td>5000</td>
<td>2573</td>
<td>466284</td>
<td>73396</td>
</tr>
<tr>
<td>MTRACK-L</td>
<td>9000</td>
<td>43865</td>
<td>199153</td>
<td>7788</td>
</tr>
<tr>
<td>MTRACK-M</td>
<td>120</td>
<td>35304</td>
<td>110759</td>
<td>1417</td>
</tr>
<tr>
<td>MTRACK-S</td>
<td>780</td>
<td>30300</td>
<td>78302</td>
<td>768</td>
</tr>
</tbody>
</table>

Alternative algorithms. While there is no established baseline or benchmark, there are algorithms we can compare with. On one extreme, we have **OptimalSparse** that should be the most effective, at the expense of speed. We experimented with **OptimalSparse** and discovered that its running time is prohibitive for the size of datasets we use; we thus omit its performance from our results. On the other extreme, there are fast heuristic methods that may yield good solutions. One such method is **SortByProbability** that sorts the arcs in \( D \) by decreasing probability (we considered a variant that sort arcs by decreasing number of actions traversing each arc, which yields worse results than **SortByProbability** and is thus omitted). Finally, we consider algorithm **Random**, that simply permutes all edges having a non-zero probability. Please note that both heuristics use **SPINE**’s first phase as an initialization procedure.

**Implementation.** All algorithms are implemented in Java using the COLT library implementation of sparse matrices and the Fastutil library for type-specific maps and collections.\(^4\) Our experiments are performed on a single Dual-\(^3\)Core 2530MHz Intel processor and using less than 10GBs of memory for the largest datasets.

For efficiency reasons we create \( F^+_{\alpha}(v) \) and \( F^-_{\alpha}(v) \) at initialization and keep them in memory to quickly compute the log-likelihood function. Most processing time during sparsification is spent accessing these data structures to compute the likelihood for a subset of the edges incident to a node.

### 6.2 Results

We execute the different sparsification algorithms and measure log-likelihood at different levels of \( k \) (Figure 4).

The plots in Figure 4 demonstrate that **SPINE** obtains sparse graphs of high or even maximum likelihood with a fraction of the total network size. For instance, we observe that for the YMEME-L dataset, the initialization procedure produces a non-zero-likelihood solution at 17% of non-zero-probability arcs – that’s about 90,000 arcs, i.e., 7% the total number of arcs. Then, **SPINE** achieves maximum likelihood at 60% of non-zero arcs – that’s about 250,000 arcs, i.e., 20% of the total number of arcs. (Note that we know **SPINE** achieves maximum likelihood for that number of arcs, because the corresponding likelihood value remains at that level for larger values of \( k \) and is equal to the likelihood of the original network). Similar observations hold for all other datasets.

\(^2\)http://snap.stanford.edu/data/memetracker9.html
\(^3\)http://acs.lbl.gov/software/colt/
\(^4\)http://fastutil.dsi.unimi.it/
The execution time for the first phase of the algorithm was about 15 minutes. For MTrack-S, the smallest dataset, the corresponding running time was less than 1 second. The time needed by Spine to obtain a sparse network of maximum likelihood is reported in Table 2 for all datasets.

<table>
<thead>
<tr>
<th></th>
<th>Spine – Time to reach max-likelihood</th>
</tr>
</thead>
<tbody>
<tr>
<td>YMemE-L</td>
<td>3.5 hrs</td>
</tr>
<tr>
<td>YMemE-M</td>
<td>35 min</td>
</tr>
<tr>
<td>YMemE-S</td>
<td>1.5 min</td>
</tr>
<tr>
<td>MTrack-L</td>
<td>10 sec</td>
</tr>
<tr>
<td>MTrack-M</td>
<td>1 sec</td>
</tr>
<tr>
<td>MTrack-S</td>
<td>&lt; 1 sec</td>
</tr>
</tbody>
</table>

7. APPLICATION: INFLUENCE MAXIMIZATION

In Section 1 we claim that sparsification is a fundamental data-reduction operation with many applications. As a proof of concept, we apply Spine as a pre-processing step for the influence-maximization problem [18], showing that computations on sparsified models give up little accuracy, but yield significant improvements in terms of scalability.

Consider a social network G with arcs annotated with probabilities of influence and assume the independent cascade model of propagation. Consider also an arbitrary set S of nodes in G (the “seed set”), and assume they are the first to perform some action α at time t = 0. Then, the spread of influence of that set is defined as the (expected) total number of nodes that are eventually influenced to perform α as well. Given an integer s, the problem of influence maximization consists in identifying a set S of cardinality no more than s that has maximum spread of influence.

We apply Spine on the original network G of YMemE-S, one of our larger datasets, to identify two sparse networks G1, G2, of k1 = 25688, k2 = 38899 arcs, respectively. Network G1 is the smallest network with non-zero likelihood identified by Spine. Network G2 is the smallest network of maximum likelihood.

For each of the aforementioned networks (G, G1, G2) we run the influence maximization greedy algorithm of [18], denoted MaxInf1, to identify seed-sets of size s, for different values of s. MaxInf1 greedily selects seed nodes, every time choosing the node that leads to the biggest increase in influence spread. To estimate influence spread, MaxInf1 performs Monte Carlo simulations of the independent cascade model. In our experiments, we perform batches of 100 simulations for every candidate seed set, and consider our estimate stable when it changes less than 10%. For each network and seed-set size s, we measure (i) the running time of the algorithm, and (ii) the influence of the identified seed set.

As shown in Figure 6, running MaxInf1 over G2 returns seed sets of almost optimal influence, at a considerable gain in speed in comparison with running on G, even for small seed sets. In addition, we observe that performing seed selection over G1, the sparsest network, leads to large gains in efficiency, while returning seed sets with high influence.

For example, when run over G, MaxInf1 identifies a seed set of size |S| = 10 and influence I0 = 290, in t0 = 54582 seconds (15 hours). On the other hand, when run over G1, it identifies a seed set of the same size and influence I1 = 0.88 · I0 = 255, in t1 = 0.09 · t0 = 5061 seconds (1.4 hours). As we can see, the gain from efficiency outweighs the loss in quality, especially for large sizes of S where running MaxInf1 on G becomes excessively expensive.
8. CONCLUSIONS AND FUTURE WORK

We study the problem of sparsifying influence networks. Given a social graph and a log of propagations, we select the $k$ arcs that best describe the propagations in terms of likelihood. We show that the problem is inapproximable within any multiplicative factor, and introduce Spine, a greedy algorithm, to solve it efficiently.

Through experimental evaluation over real datasets, we demonstrate that Spine identifies sparse sub-networks with practically little compromise in quality. We demonstrate examples of such sparse subnetworks, and apply sparsification as a pre-processing step to the influence maximization problem, showing that it provides significant gains in efficiency at little loss of quality.

Given the decomposition of the problem and hence its suitability to parallelization (see Section 5.2, ‘A note on parallelization’), in our on-going work we are developing a Hadoop implementation of Spine, which will allow to scale to extremely large networks.

Reproducibility. Code implementing the Spine algorithm, along with instructions to repeat the experiments over the public MTrack data source, is available at http://queens.db.toronto.edu/~mathiou/spine/.

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