Coming to Our Senses: Reconnecting Mathematics Understanding to Sensory Experience

Ana Pasztor[1]
School of Computer Science
Florida International University
University Park, Miami, FL 33199
pasztora@cs.fiu.edu

Mary Hale-Haniff
School of Social and Systemic Studies
Nova Southeastern University
3301 College Avenue
Fort Lauderdale, FL 33314
hale-haniff@email.msn.com

and

Daria M. Valle
Claude Pepper Elementary School
14550 S.W. 96 St.
Miami, FL 33186

The child cannot conceive of tasks, the way to solve them and the solutions in terms other than those that are available at the particular moment in his or her conceptual development. The child must make meaning of the task and try to construct a solution by using material she already has. That material cannot be anything but the conceptual building blocks and operations that the child has assembled in his or her own prior experience.

von Glasersfeld (1987, p. 12)

Introduction

At this time, we are experiencing a global shift from a positivist (rationalist) paradigm toward a constructivist (naturalistic) paradigm. This shift is emerging in a wide range of academic areas such as philosophy, the arts, education, politics, religion, medicine, physics, chemistry, ecology, evolution, psychology, linguistics (Lincoln & Guba, 1985; Schwartz & Ogilvy, 1979), and mathematics—mathematics education in particular.

The term "paradigm" refers to a systematic set of assumptions or beliefs that comprise our philosophy and world view. Beginning with fundamental ideas about the nature of knowing and

[1] This work was partially supported by the following grants: NSF-CISE-EIA-9812636 with the DSP Center, NSF-MII-EIA-9906600 with the CATE Center, and ONR-N000 14-99-1-0952 at Florida International University.

understanding, paradigms shape what we think about the world (but cannot prove). Our actions in the world, including the actions we take as inquirers, cannot occur without reference to those paradigms (Lincoln & Guba, 1985). In mathematics education the paradigm shift has been a top-down shift beginning with the theoretical foundations of mathematics education and then moving to the level of professional organizations which have been leading extensive efforts to reform school mathematics according to constructivist principles (National Council of Teachers of Mathematics—NCTM, 2000; National Science Foundation—NSF, 1999).

The new 2000 Principles and Standards for School Mathematics of the National Council of Teachers of Mathematics (NCTM, 2000) may be the most significant effort up to this time. So far, however, the paradigm shift is not yet emanent at the grass roots level of the classroom in terms of actual changes in mathematics classroom practices. One of the reasons for this may be that the constructivist theories espoused by the researchers are as yet too abstract to readily lend themselves to implementation. Even NCTM's (2000) new guidelines, which were designed to provide "focused, sustained efforts to improve students' school mathematics education" (NCTM, 2000, chapter 1) do not translate readily into classroom practice. However, this is to be expected, given that the very same communities whose members started the constructivist reform movement often lack an awareness for the need to translate the new principles even to their own behavior, let alone to embody them. "This is not altogether surprising because leading practitioners at all levels tend to be so busy with day-to-day problems that they seldom have adequate time for metalevel considerations. As the folk saying states: 'When you are up to your neck in alligators, it's difficult to find time to think about draining the swamp'" (Lesh, Lovitts, & Kelly, 1999, p. 32).

In this paper we will describe an ongoing *pilot project* in elementary mathematics education aimed at exploring the following two of the six NCTM (2000) principles for school mathematics:

> **Teaching**. Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well.
>
> **Learning**. Students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge. (NCTM, 2000, chapter 2).

Careful reading of these two standards raises a number of questions: What is mathematics knowledge? What constitutes understanding? What is learning with understanding? How do we gain access to students' experience and prior knowledge? What *kind* of experience and prior knowledge do we want our students to build their new knowledge from? How can the teacher make sure that she is helping the students build from *their own* experience, rather than from what happens to be the teacher's experience?

These and related questions drive our pilot research project. The pilot project, in turn, is part of a larger, ongoing project that we have come to call the *Linguistic Action Inquiry Project*. The goal of the Linguistic Action Inquiry Project has been to facilitate change in a variety of domains of human communication. Its primary tool has been the utilization and refinement of a *shared experiential language* (SEL) and the enhancement of *the person* of the facilitator, be that a teacher, a therapist, or a researcher, as the main work instrument. Our pilot project is an application of the Linguistic Action Inquiry Project. Its goal is twofold:

1. to investigate how the methodology of linguistic action inquiry can help successfully root mathematical understanding in students' prior sensory experiences, and
2. to learn, utilizing SEL, how students naturally organize their experiences when they try to understand mathematics.

In all action-research cycles of the pilot project we utilize SEL to model, help adjust fit of and reflect upon students' experiences linked to mathematics understanding. This represents a unique opportunity to document such experiences for the purpose of refining mathematics teaching methodologies and curricula in ways that allow consistent understanding to become attainable by every citizen, not just a few "elite."

To describe the pilot project, we have structured our paper as follows: First, we will lay the groundwork of our guiding theoretical framework by contrasting positivist and constructivist paradigms and their methodological implications both for teaching and learning in general, and for mathematics education in specific. We will then present existing efforts to demystify mathematics and reconnect it to students' everyday experiences, and we will argue for the need to root consistent mathematics understanding in students' *sensory* experiences. We conclude the first part of the paper by defining basic components of SEL, the shared experiential language, which is the prerequisite for both our Linguistic Action Inquiry Project and the pilot project. In the second part of the paper we will describe our constructivist linguistic action inquiry methodology, where the person of the educator/researcher is the primary teaching/research instrument, and which we have been developing in the context of our Linguistic Action Inquiry Project. In the third part of the paper we then illustrate our pilot project where we are adapting this approach to teaching mathematical thinking to a group of fourth-grade students in a manner that effectively implements the intent of the NCTM 2000 guidelines. Lastly, we offer some concluding thoughts and suggestions.

1. The Guiding Theoretical Framework

1.1 Contrasting Positivist and Constructivist Paradigms in Education

Positivist and constructivist paradigms can be contrasted in terms of differences in ontology (assumptions regarding the nature of reality), epistemology (assumptions about how we know what we know), and methodology (Denzin & Lincoln, 1994; Lincoln & Guba, 1985). **Table 1** summarizes some key distinctions between the two thought systems as they relate to our subsequent discussions on mathematics education. It also parallels table 2.1 of (Kelly & Lesh, 1999, pp. 37-38), particularly from the point of view of methodology.

	Positivist View	**Constructivist View**
Nature of Reality and Knowledge	Reality is a single and fixed set of knowable, objective facts to be discovered.	Reality is not accessible. Multiple and dynamic subjective constructions and interpretations are possible.
	Reality is fragmentable into pieces which can be studied in isolation.	Aspects of knowledge can only be understood in relationship to the larger context.
	Knowledge is matching reality	Knowledge is finding *fit* with observations.
Nature of the Learning/Teaching Process	Teaching is one-way transmission of fixed knowledge to the passive student.	Teacher and student both actively participate to co-create new learnings.

Nature of Perception	The basic unit of perception is singular, objective truth. People internalize information.	The basic unit of perception is linguistic and social. Knowledge is an interaction of people and ideas; a process of communication where people co-create experience together
Role of Values	Both the teacher and what is being taught are objective and value-free.	Both the teacher and what is being taught are subjective and value-bound.
Relationship between Knowledge and the Knower	Separate, dualistic, hierarchical	Inseparable, mutually-engaging, cooperative
Goal of Teacher Training	Enhance content and presentation of *information*	Enhance the *person-of- the- teacher* as primary teaching instrument
Measures of Understanding	Focus on replication of content: finding the correct answer or end result	Focus on process of understanding
	Attends mainly to auditory-verbal aspects of student communication	Attends to multi-sensory aspects of communication including presenting emotional state and conceptual experience of students
	Focuses on conscious, auditory, literal ways of knowing	Focuses on both conscious and other-than-conscious and interpretive ways of knowing
	Attends primarily to the content of the unitary concept being taught	Attends to the holistic presuppositional system of related knowledge
	Emphasizes finding a *match* with conventional responses	Emphasizes *fit* with experience
Standards for Comparison	Normative (self-to-other) comparisons with external references derived from quantitative data	Emphasizes self- to-self comparisons and self-to-other comparisons derived from qualitative data
Teaching of Abstractions	Attempts to teach abstractions in isolation from sensory-based experience.	Abstractions are embodied, sensory-based concepts.
	Particular constructs are taught without regard to how they fit with the whole system of constructs and unifying metaphors.	Integrates particular learnings with system of relatationships among concepts; use of metaphors is congruent with a unified system of abstractions

Mode of Inquiry	Primarily quantitative	Qualitative and mixed designs
Criteria for Inquiry	Reliability and validity	Meaning and usefulness

Table 1. Contrasting implications of positivist and constructivist assumptions for education.

1.2 The traditional view of knowledge and its implications to mathematics

In this subsection we discuss the positivist view of knowledge, its paradoxical nature, the view of mathematics as the purest form of reason, and implications to the educational system.

The traditional, positivist approach to instruction has been referred to as "the age of the sage on the stage" (Davis & Maher, 1997, p. 93), due to its "transmission" model of teaching, where teaching means "getting knowledge into the heads" of the students (von Glasersfeld, 1987, p. 3), that is, *transmitting* knowledge from the teacher to the student. The underlying philosophy is that knowledge is out there, independent of the knower, ready to be discovered and be transferred into people's heads. It is "a commodity that can be communicated" (von Glasersfeld, 1987, p. 6). The *ontology* presupposed in this view is that there is one true *reality* out there, which exists independently of the observer. Furthermore, we have access to this reality, and we can fragment, study, predict and control it (Lincoln & Guba, 1985; Hale-Haniff & Pasztor, 1999).

However, as von Glasersfeld (1987) points out, while trying to access reality, we have been caught in an age long dilemma: On one hand truth is (traditionally) defined as "the prefect match, the flawless representation" of reality (von Glasersfeld, 1987, p. 4), but on the other hand, we *all* live in a world of genetic, social and cultural constraints, some of which none of us can ever "escape." W*ho then, is to judge "the perfect match with reality"?*

To answer this question, Western philosophy has overwhelmingly made the assumption that given the right tools, pure reason is able to transcend all constraints and the confines of the human body, including those of perception and emotion. In traditional Western philosophy mathematical reasoning has been seen as the purest example of reason: "purely abstract, transcendental, culture-free, unemotional, universal, decontextualized, disembodied, and hence formal" (Lakoff & Nuñez, 1997, p. 22). Mathematics was seen to be "just out there in the world—as a timeless and immutable objective fact—structuring the physical universe" (Lakoff & Nuñez, 1997, p. 23). One of the best examples of this powerful objectivist view of mathematics is Platonism, a view held by most great mathematical minds even of our century, including Albert Einstein, Kurt Gödel, and Roger Penrose, a view that a *unique* "correct" mathematics exists "out there" independent of any minds in some "Platonic realm—the realm of transcendental truth." But as Lakoff (1987, chap. 20, pp. 355-361) has shown, even within an objectivist stance Platonism runs into problems, being incompatible with the so-called independence results of mathematics. Without going into its details, here is a brief description of Lakoff's arguments: 1. The so-called Zermelo-Fraenkel axioms plus the axiom of choice (ZFC axioms in short) characterize set theory in a way that all branches of mathematics can be defined in terms of set theory; 2. There exist two extensions of ZFC, let us call them ZFC1 and ZFC2 for our purposes here, as well as a mathematical proposition *P*, such that *P* is true in a model of ZFC1, but is false in a model of ZFC2. This means that *P* is independent of ZFC, and ZFC1 and ZFC2 define two different mathematics; 3. If ZFC defines a mathematics that is transcendental, then so do ZFC1 and ZFC2; 4. We conclude that even if the mathematics defined by ZFC is transcendental, *it cannot be unique*.

The goal of the traditional scientist, mathematician, or, in general, researcher, is to find objective truth. Thus, she is trained to be value-neutral in order to be able to objectively judge "the perfect match" with reality. In practice, however, there is a direct "relationship between claims to truth and the distribution of power in society" (Gergen, 1991, p. 95). This is no different in education. Gergen (1991) argues that "because our educational curricula are largely controlled by 'those who know,' the educational system operates to sustain the existing structure of power. Students learn 'the right facts' according to those who control the system, and these realities, in turn, sustain their positions of power. In this sense the educational system serves the interests of the existing power elite" (p. 95). Those at the top of the educational system hierarchy are the "objective" experts of knowledge, they determine teaching goals and criteria of assessment. Accordingly, the teacher-student relationship is also a hierarchical, authoritarian relationship.

Although there "is a growing rejection of the researcher as the expert—the judge of the effectiveness of knowledge transmission" (Kelly & Lesh, 1999, p. 39), the myth of objectivity has been holding up very well in mathematics and science, partly because the idea of objectivity "is seductive in its apparent simplicity and clarity: Whoever succeeds in comprehending nature's intrinsic order, in its existence independent of human opinions, convictions, prejudices, hopes, values, and so on, has eternal truth on his side" (Watzlawick, 1984, p. 235). However, problems arise when a system claims possession of absolute truth and consistency. As it is unable to prove its truth and consistency from within, it has to revert to authority: "[T]he concept of an ultimate, generally valid interpretation of the world implies that no other interpretations can exist beside the one; or, to be more precise, no others are permitted to exist" (Watzlawick, 1984, p. 222).

If objectivity of mathematics is just a myth, one may ask, what happens to basic *facts* such as "two and two is four?" Are we denying them? Absolutely not! However, we hold the view that they are created by us humans (hence the origin of the word "fact" in "factum," meaning "a deed" in Latin—c.f. (Vico, 1948)). For example, counting presupposes that we group things together to count them. Groupings are not out there in the world, independent of us. Grouping things together and counting them are characteristics of living beings, not of an external reality (Lakoff & Nuñez, 1997). Numbers, then, are concepts that we use to communicate about our shared experiences as a species. More generally, mathematics is not the study of transcendent entities, but "the study of the structures that we use to understand and reason about our experience—structures that are inherent in our preconceptual bodily experience and that we make abstract via metaphor" (Lakoff, 1987, pp. 354-355).

1.3 The constructivist view of knowledge and its implications to mathematics education

In contrast to positivist philosophy, constructivist philosophies have adopted a concept of knowledge that is *not* based on any belief in an accessible objective reality. In the constructivist view, knowing is not matching reality, but rather finding a *fit* with observations. Constructivist knowledge "is knowledge that human reason derives from experience. It does not represent a picture of the 'real' world but provides structure and organization to experience. As such it has an all-important function: It enables us to solve experiential problems" (von Glasersfeld, 1987, p. 5). With this theory of knowledge, the experiencing human turns "from an explorer who is condemned to seek 'structural properties' of an inaccessible reality … into a builder of cognitive structures intended to solve such problems as the organism perceives or conceives" (von Glasersfeld, 1987, p. 5).

Traditional views of reason as disembodied and objective, mind as a symbol-manipulating machine, and intelligence as computation (Simon, 1984; Minsky, 1986; Dennett, 1991) have given way to a more contemporary view of reason as "embodied" and "imaginative" (Lakoff, 1987, p. 368) and inseparable from our bodies; mind as an inseparable aspect of physical experience (Damasio, 1994; Pert, 1997; Varela, Thomson, & Rosch, 1991):

> Human concepts are not passive reflections of some external objective system of categories of the world. Instead they arise through interactions with the world and are crucially shaped by our bodies, brains, and modes of social interaction. What is humanly universal about reason is a product of the commonalities of human bodies, human brains, physical environments and social interactions." (Lakoff & Nuñez, 1997, p. 22).

For the constructivist-informed educator, the process of facilitating mathematical understanding is a process of co-construction of multiple meanings in which she accommodates her own mathematical understanding to fit with resourceful elements of the students' own experiences. It is a process that leads to "a viable path of action, a viable solution to an experiential problem, or a viable interpretation of a piece of language", and "there is never any reason to believe that this construction is the only one possible" (von Glasersfeld, 1987, p. 10).

In constructivism, the meaning of learning has shifted from the student's "correct" replication of what the teacher does to "the student's *conscious understanding* of what he or she is doing and why it is being done" (von Glasersfeld, 1987, p. 12):

> Mathematical knowledge cannot be reduced to a stock of retrievable 'facts' but concerns the ability to compute new results. To use Piaget's terms, it is *operative* rather than *figurative*. It is the product of reflection—and whereas reflection as such is not observable, its product *may* be inferred from observable responses." (von Glasersfeld, 1987, p. 10)

The term "reflection" refers to the ability of the mind to observe its own activity. Operative knowledge, on the other hand, refers to the ability to know what to do to construct a solution, as opposed to giving a conditioned response. Operative knowledge is constructive. "It is not the particular response that matters but the way in which it was arrived at" (von Glasersfeld, 1987, p. 11).

But how is the student to attain such operative knowledge in mathematics, when the "structure of mathematical concepts is still largely obscure" (von Glasersfeld, 1987, p. 13)? Most definitions in mathematics are *formal* rather than *conceptual*. In mathematics, definitions "merely substitute other signs or symbols for the definiendum. Rarely, if ever, is there a hint, let alone an indication, of what one must *do* in order to build up the conceptual structures that are to be associated with the symbols" (von Glasersfeld, 1987, p. 14). To mend the situation, recently mathematics education researchers have been redefining mathematical concepts as imagery, metonymy, analogy, and metaphor (English, 1997) to open up new possibilities for operative understanding rooted in the students' own experiences. In the next section we present some of these and other recent efforts to reconnect mathematical understanding to students' prior experiences.

1.4 Mathematical abstraction as metaphorical structure rooted in subjective experience

Abstract mathematical concepts, just as abstract concepts in general, are *metaphorical* and are built from people's sensory experiences (Lakoff & Nuñez, 1997; Lakoff & Johnson, 1999). Therefore,

> teaching mathematics necessarily requires teaching the metaphorical structure of mathematics. This should have the beneficial effect of dispelling the myth that mathematics is literal, is inherent in the structure of the universe, and exists independent of human minds. (Lakoff & Nuñez, 1997, p. 85)

Abstract mathematical ideas are almost always defined by metaphorical mappings from concrete, familiar domains. Understanding takes place when these concrete domains *fit* the students' *own, individual experience*, and frustration and confusion ensues when they are incongruent. English (1997) provides a very good example of what happens if the metaphorical mapping is rooted in an a-priori construction that doesn't *fit* the students' own individual experience. The example concerns the use of a line metaphor to represent our number system, whereby numbers are considered as points on a line.

The "number line" is used to convey the notion of positive and negative number, and to visualize relationships between numbers. It turns out that students frequently have difficulty in abstracting mathematical ideas that are linked to the number line (Dufour-Janvier, Bednarz, & Belanger, 1987, quoted in English, 1997, p. 8). "There is a tendency for students to see the number line as a series of 'stepping stones,' with each step conceived of as a rock with a hole between each two successive rocks. This may explain why so many students say that there are no numbers, or at the most, one, between two whole numbers" (English, 1997, p. 8).

This example also serves as an excellent demonstration of the notion of "*fit*" as opposed to "match." The student's own representation of the "number line" *fits* the purpose it has to serve only as long as the constraints in the environment conform to it. When the student hits obstacles in "understanding," she needs to adjust the *fit* of his or her representation, or learning will be impeded.

Sometimes, the students have the necessary resources and are able *to adjust the fit themselves*. An example offered by Davis and Maher (1997, pp. 101–102) illustrates this. The students in this example have 12 meters of ribbon. As part of a more complex problem, they have to determine how many bows they can make if each bow requires two thirds of a meter of ribbon. Previously the children have determined that they were able to make 36 bows from a single 12-meter package of ribbon if one bow required one third of a meter. At this point they took their previous answer for one third of a meter bows and doubled it, concluding they would be able to make 72 bows. However, one of the students objected that it made no sense that they were getting *more* bows from a single 12-meter package of ribbon when each individual bow was *larger* than in the previous case. It made sense to get more bows if the individual bows were smaller, but not if they were larger. The children then re-worked their answer to get one that *fit* their experience.

While in the previous example the students were able to reorganize their own experience in a way that made it *fit* the constraints of the problem at hand, often times the teacher needs *to provide for the students* "precisely those experiences that will be most useful for further development or revision of the mental structures that are being built" (Davis & Maher, 1997, p. 94). This idea is wonderfully demonstrated by Machtinger (1965) (quoted in Davis & Maher,1997, pp. 94–95) who taught kindergarten children to conjecture and prove several theorems about numbers, including even+even=even, even+odd=odd, and odd+odd=even. She did so by defining a number *n* as "even" if a group of *n* children could be organized into pairs for walking along the corridor and as "odd" if such a group had one child left over when organized into pairs. Since walking along the corridor in pairs was a daily experience for the children, learning the new information became a matter of just expanding or reorganizing their existing knowledge.

However, expanding or reorganizing existing knowledge is not always possible. As we saw in the number line example, understanding is not possible where a teacher has inadvertently used incompatible metaphors to explicate mathematical ideas. To examine this phenomenon in more detail, let us consider the so-called *grounding metaphors* defined by Lakoff and Nuñez (1997). Grounding metaphors ground mathematical ideas in everyday experience. Three of such grounding metaphors are listed and discussed below.

- *Arithmetic Is Object Collection*. Restrictions of this metaphor are, for example, the following: Numbers Are Collections of Physical Objects of uniform size, Arithmetic Operations Are Acts of Forming a collection of objects, The Size of the Number Is the Physical Size (volume) of the collection, The Unit (One) Is the Smallest Collection, Zero Is An Empty Collection. Here are some linguistic manifestations of this metaphor: "There are 4 5's *in* 23, and 3 *left over*." "How many *more* than 5 is 8? 8 is 3 *more* than 5." "7 is too *big* to *go into* 10 more than once." (Lakoff & Nuñez, 1997, p. 36).
- *Arithmetic Is Object Construction*. Some restrictions of this metaphor are, for example, the following: Numbers Are Physical Objects, Arithmetic Operations Are Acts of object construction, The Unit (One) Is the Smallest whole object, Zero Is the Absence of Any Object. Here are some linguistic manifestations of this metaphor: "If you put 2 and 2 together, it *makes* 4." "What is the *product* of 5 and 7?" "2 is a *small fraction* of 248." (Lakoff and Nuñez, 1997, p. 36).
- *Arithmetic Is Motion*. Some restrictions of this metaphor are, for example, the following: Numbers Are Locations on a Path, Arithmetic Operations Are Acts of Moving along a path, Zero Is The Origin, The Smallest Whole Number (One) Is A Step Forward from the origin. Here are some linguistic manifestations of this metaphor: "How *close* are these two numbers?" "4.9 is *almost* 5." "Count up to 20, without *skipping* any numbers." "Count *backwards* from 20." (Lakoff & Nuñez, 1997, p. 36).

The teacher who uses the Collection and Construction metaphors to define the natural numbers will run into problems because these metaphors don't usually work for defining negative numbers, rational numbers, or the reals in a way that leads to consistent understanding. For example, a teacher might want to teach the equation (-1) + (-3) = (-2). He might, for this purpose, extend the Object Collection metaphor by the metaphor Negative Numbers Are Helium Balloons, and use it together with Quantity is Weight and Equations are Scales. As helium balloons are seen as having negative weight, they offset positive weight on the scale. However, as Lakoff and Nuñez (1997) put it, "[t]his ad hoc extension will work for this case, but not for multiplying by negative numbers. In addition, it must be used with care, because it has a very different cognitive status than the largely unconscious natural grounding metaphor. It cannot be added and held constant as one moves to multiplication by negative numbers" (Lakoff & Nuñez, 1997, p. 39).

Whether consciously or unconsciously, every teacher uses metaphors to teach mathematical ideas. If used consciously and with care, however, metaphors can become a tool to facilitate consistent understanding.

1.5 Consistent Understanding: the need to root it in sensory experience

Consistent understanding is the key to successful mathematics learning. But just *what is consistent understanding?* In trying to answer this question, let us start with the classroom practice, where we can detect whether or not such understanding is taking place. In practice, "[f]or too many people, mathematics stopped making sense somewhere along the way. Either slowly or dramatically, they gave up on the field as hopelessly baffling and difficult, and they grew up to be adults who—confident that others share their experience—nonchalantly announce, 'Math was just not for me' or 'I was never good at it.'" (Askey, 1999). It has become "socially acceptable to dislike and be unsuccessful at mathematics" (Doerr & Tinto, 1999, p. 423)—you either have the "math genes" or you don't. Many clients, when they see Hale-Haniff in her psychotherapy practice, tell her that they would have chosen another path in life if only they had been able to understand math. And too many people, upon hearing that Pasztor is a mathematician, confess, after a sigh of awe, that they either “hated” math or their mathematics teacher.

Ruth McNeill (1988) shares her story of how she came to quit math: “What did me in was the idea that a negative number times a negative number comes out to a positive number. This seemed (and still seems) inherently unlikely—counterintuitive, as mathematicians say. I wrestled with the idea for what I imagine to be several weeks, trying to get a sensible explanation from my teacher, my classmates, my parents, anybody. Whatever explanation they offered could not overcome my strong sense that multiplying intensifies something, and thus two negative numbers multiplied together should properly produce a *very* negative result” (McNeill, 1988—quoted in Askey, 1999).

What Ruth’s mathematics teacher must have failed to recognize was that there was a very strong negative experience forming as a result of Ruth no being able to resolve the incongruity between her internalized metaphor “Multiplication Intensifies,” and what she was being told by her teacher. Ruth dealt with this dissonance by pretending “to agree that negative times negative equals positive … [u]nderneath, however, a kind of resentment and betrayal lurked, and” she “was not surprised or dismayed by any further foolishness” her “math teachers had up their sleeves … Intellectually,” she “was disengaged, and when math was no longer required,” she “took German instead” (McNeill, 1988—quoted in Askey, 1999).

In order to find the roots of such widely experienced frustrations with mathematics, let us take a closer look at the concept of mathematics understanding. In mathematics education research, the following is the still predominant definition of understanding: “A mathematical idea or procedure or fact is understood if it is part of an internal network. More specifically, the mathematics is understood if its mental representation is part of a network of representations. The degree of understanding is determined by the number and the strength of the connections. A mathematical idea, procedure, or fact is understood thoroughly if it is linked to existing networks with stronger or more numerous connections” (Hiebert & Carpenter, 1992, p. 67). Knowledge structures and semantic nets have been used to implement the concept of mental representations and their connections (for references see Hiebert & Carpenter, 1992, p. 67; in addition, see Thagard, 1996). More recently, in research on mathematics reasoning, particular attention has been given to the knowledge structures of analogy, metaphor, metonymy, and images (English, 1997). These structures, or rather, constructs, play a powerful role in mathematics learning—a role that “has not been acknowledged adequately. Given that ‘Mathematics as Reasoning’ is one of the curriculum and evaluation standards of the National Council of Teachers of Mathematics (USA), it behooves us to give greater attention to how these vehicles for thinking can foster students’ mathematical power” (English, 1997, p. viii).

However, sensory representations such as visual, auditory, or kinesthetic images (c.f. Damasio, 1994) are, in a Batesonian (1972) sense, knowledge structures of a different “logical level” than analogies, metaphors, or metonymy (Thagard, 1996; English, 1997). For example, according to (Lakoff & Johnson, 1999), a metaphorical idiom is “the linguistic expression of an image plus knowledge about the image plus one or more metaphorical mappings. It is important to separate that aspect of the meaning that has to do with the general metaphorical mapping from that portion that has to do with the image and knowledge of the image” (p. 69). Indeed—a person may represent a metaphor in either sense system: visually, auditorily, or kinesthetically.

To be able to help students attain consistent mathematical understanding and to be able to recognize when it takes place, we need to *retrace knowledge structures*, be they metaphors, metonymy, analogy, or concepts, to their sensory components, which, as we shall see in the next section, are precisely images of various sensory modalities.

A great deal of discussion has also been devoted to the question of how to help students make new connections in their network of representations. Should it be *bottom-up,* where instruction builds on students' prior knowledge, or should it be *top-down*, where instruction starts with the kind of connections that the expert makes and works backwards to teach the students to make the same kinds of connection (Hiebert & Carpenter, 1992; Cobb et al., 1997)?

These discussions of understanding mathematics have come a long way from the "transmission" model of positivism. In fact, recently there has been a move away from a largely disembodied approach rooted in "first generation" cognitive science or philosophical cognitivism (Lakoff & Johnson, 1999) towards "reasoning with structures that emerge from our bodily experiences as we interact with the environment" and that "extend beyond finitary propositional representations (Johnson, 1987)" (English, 1997, p. vii and p. 4). What we propose in this paper is to go one step further and deconstruct these structures and their connections into their *sensory-based components*, so that understanding abstract mathematical concepts becomes a matter of accessing one's own sensory experiences.

Our overall aim is make mathematics accessible to every single student in the classroom, as opposed to only a few "elite." "Reform," after all, "must differentiate between *expertise* in mathematics and science and *elitism*, and make expertise an accessible goal for all citizens" (Confrey, 1999, p. 93).

1.6 The shared experiential language SEL: its see/hear/feel components

According to Damasio (1994), at each moment in time our subjective experience is manifested in what he calls an "image": a *visual image*, that is, an internal picture; an *auditory image*, that is, sounds—discrete or analog; a *kinesthetic image*, that is, a feeling or an internal smell or taste; or a combination of these. For example, while J's representation of "even number" is manifested in a fuzzy visual image of the number two, accompanied by "a feeling of 2ness," and Ana's representation is a sharp visual image of 2n, written in white on a blackboard and situated right in front of her, Mary represents "even number" by hearing the actual definition of "even number."

Many people argue that they don't think in images, but rather in words or abstract symbols. But "most of the words we use in our inner speech, before speaking or writing a sentence, exist as auditory or visual images in our consciousness. If they did not become images, however fleetingly, they would not be anything we could know. This is true even for those topographically organized representations that are not attended to in the clear light of consciousness, but are activated covertly" (Damasio, 1994, p. 106).

Damasio (1994) goes as far as to require as an essential condition for having a mind the ability to form internal (visual, auditory, kinesthetic) images, and to order them in the process we call thought. His view is that "having a mind means that an organism forms neural representations which can become images, be manipulated in a process called thought, and eventually influence behavior by helping predict the future, plan accordingly, and choose the next action" (p. 90).

As we have seen, a great number of authors in constructivist research on mathematics education have recently become concerned with the "mental representations" that children build in their heads, but these authors fail to specify *what exactly these representations are in terms of our full bodily experiences*. Von Glasersfeld (1987), for example, refers to mental representations quite vaguely as "conceptions." But he makes the very important point that

> in the constructivist view, "concepts," "mental representation," "memories," "images," and so on, must not be thought of as static but always as *dynamic*; that is to say, they are not conceived as postcards that can be retrieved from some file, but rather as

> relatively self-contained programs or production routines that can be called up and run [c.f. Damasio's (1994) dispositional representations]. Conceptions, then, are produced internally. They are replayed, shelved, or discarded according to their usefulness and applicability in experiential contexts. The more often they turn to be viable, the more solid and reliable they seem. But no amount of usefulness or reliability can alter their internal, conceptual origin. They are not replicas of external originals, simply because no cognitive organism can have access to 'things-in-themselves' and thus there are no models to be copied. (von Glasersfeld, 1987, p. 219)

What does it mean then, from our perspective, when we talk about representations? Before we answer this question, we need to distinguish *perceptual* images, such as when I see a car (visual), hear a voice (auditory), or feel the chair on which I sit (kinesthetic), from *recalled* images, that can be remembered or constructed, such as when I remember a car that I have seen or imagine a car that I would like to own (visual); I remember my mother's voice or compose a new musical piece (auditory); or I remember the touch of the cold water on my toe or imagine how it would feel to shake hands with E.T. (kinesthetic).

Then recalled images of, say, an event X, *represent* X if they are able to produce in the experiencing person a reconstruction of the kind of experience he has come to call "X." Although it is of the same "kind," the experience that a person has come to call "X" is most of the time different from the experience that a representation of X triggers (von Glasersfeld, 1987). In this way a person is able to distinguish "reality" from imagination.

While we represent our experiences, as we have seen, in *all* of our senses, traditionally teachers are (implicitly) trained to teach only to the verbally oriented conscious mind, and so they often ignore visual and kinesthetic aspects of experience, thus ignoring communications related to intra-personal, emotional, and unconscious experience. However, if we intend to use experience in a holistic manner *engaging all of our senses*, we need to also honor other ways of communicating:

> For the constructivist teacher—much like the psychoanalyst—'telling' is usually not an effective tool. In this role, the teacher is much less a lecturer, and much more of a coach (as in learning tennis, or in learning to play the piano). A recent slogan describes this by saying 'the Sage on the Stage has been replaced by the Guide on the Side.' It is the *student* who is doing the work of building or revising [… his or her] personal representations. The student builds up the ideas in his or her own head, and the teacher has at best a limited role in shaping the student's personal mental representations. The experiences that the teacher provides are grist for the mill, but the student is the miller. (Davis & Maher, 1997, p. 94)

Having (we hope successfully) argued for the need to reconnect mathematics understanding to our sensory experiences, and having discussed the see/hear/feel components of such experiences, we will now turn to the second part of our paper, in which we will describe our constructivist methodology, where the person of the educator/researcher is the primary teaching/research instrument and which we have been developing in the context of our Linguistic Action Inquiry Project.

2. Methodology

We have seen that there is a general agreement across the constructivist research in mathematics education that for consistent understanding to happen, new knowledge has to attach to students' prior experiences. But we need to ask, what *kind* of prior experiences? Which ones are optimal for new learnings? How can an investigator/teacher behave in a way as to resurrect those experiences? What are resource states of learning? How is attention configured when participants are in resourceful compared to unresourceful states of consciousness? How can an investigator/teacher know when s/he is eliciting an unuseful experience? Even though people's subjective experiences are private, can students and teachers come to share a language of experience? How?

These and related questions guide our research in our pilot project, the overall goal being to successfully root students' mathematical understanding in their prior sensory experience. The pilot project involves a group of fourth grade children and three teachers/investigators who, for several years have been engaged in a larger action research project informed by what we have come to call constructivist linguistic action inquiry. Over the past fifteen years, this action research group has been a format with an overall purpose to facilitate change in a variety of domains of human communication. Its primary tool has been the utilization and refinement of the *shared experiential language*, SEL, and the enhancement of *the person* of the facilitator, be that a teacher, a therapist, or a researcher, as the main work instrument. Since our methodology presupposes that teachers are researchers and researchers are teachers, we refer to the facilitators as the teachers/investigators or some variation thereof.

2.1 Teacher/investigator participants: A multidisciplinary perspective

The three authors bring a multidisciplinary perspective to both our Linguistic Action Inquiry Project and the pilot project: Pasztor's expertise is mathematics and cognitive science (she does the mathematics teaching in the pilot project, assisted by Valle), Valle's expertise is elementary special education, school counseling, and family therapy (—she is the science teacher of the fourth grade class in the project), and Hale-Haniff's expertise is training the person-of-the practitioner/researcher in the fields of constructivist psychotherapy, education, and business.

Accordingly, our choice of research methods has been influenced by recent developments in all of the disciplines we represent.

Educational research, as we have seen, has gradually moved from a positivist to a *constructivist* paradigm. This move has been reflected in increased use of qualitative and mixed design studies and in efforts to re-examine methodologies for congruence with the espoused philosophy. Such efforts are reflected in a recent a Workshop on Research Methods held by the National Science Foundation with the goal to establish new "guiding principles for designing research studies and evaluating research proposals of mathematics and science education" that utilize "alternative methods for research" (NSF, 1999). The most important "alternative methods for research" are discussed in (Kelly & Lesh, 1999), particularly those that "radically increase the relevance of research to practice" (Lesh, Lovitts & Kelly, 1999, p. 18).

A different, equally important shift has taken place in *cognitive science*. This shift is set against the backdrop of the legacy of behaviorism that tried hard to do away with the study of people's "murky interiors." While even mentioning subjective experience was, for decades, a taboo, an exponentially growing number of recent publications in cognitive science have been concerned with the scientific study of subjective experience. Chalmers' (1995) seminal paper set a new direction in cognitive science research, drawing on a great number of publications and giving rise to even more on the so-called "hard problems of consciousness" (Churchland & Sejnowski, 1992; Crick, 1994; Baars,1988; Calvin, 1990; Dennett, 1991; Edelman, 1989; Jackendoff, 1987; Nagel,1986; McGinn, 1991; Chalmers, 1996; Flanagan, 1992; Globus, 1995; Johnson, 1987; Lakoff & Johnson, 1999; Searle, 1992; Varela, 1996; Varela, 1996a), which basically situates the study of subjective experience *outside* of the scope of standard methods of cognitive science, whereby phenomena are explained in terms of computational or neural mechanisms.

While the field of cognitive science seems "stuck" on questions such as whether it is possible for a third person to know a first person's subjective experience, the field of *constructive therapy* is not only able to get a handle on subjective experience, but is able to do so in a manner that affects people deeply, helping them change in ways they find useful (Hoyt, 1994; Neimeyer & Mahoney, 1995; Hale-Haniff & Pasztor, 1999).

In exploring the different approaches to subjective experience in cognitive science, Hale-Haniff and Pasztor (1999) noted that the assumptions and methodologies of these approaches were based primarily on a positivist paradigm. Viewing the "hard problem of consciousness" through a constructivist lens, it became clear to the authors that the positivist paradigm, by virtue of its assumptions that knower and known are separate and uninfluenced by each other, *a priori* situates the study of subjective experience outside the limits of what can be known. By contextualizing the study of subjective experience within the constructivist epistemology, ontology, and methodology, Hale-Haniff and Pasztor (1999) were able to ask new questions regarding co-created, subjective experience, questions that could not have arisen within the positivist thought system. In our projects and subsequent research, we follow the direction set by (Hale-Haniff & Pasztor, 1999), working with *methods of qualitative inquiry*, where *the self of the investigators is the major research instrument*.

2.2 How subjectivity plays out in our research

A main characteristic of qualitative inquiry is that the researcher does not rely as much on propositional knowledge, but more on his or her tacit knowledge about the nature of human experience. She holds in her physiology the patterns of human behavior, so that she is predisposed to notice the finest clues in other's behavior. The researcher and her research instrument are one. The human instrument "uses methods that are appropriate to humanly implemented inquiry: interviews, observations, document analysis, unobtrusive clues, and the like" (Lincoln & Guba, 1985, pp. 187-188). However, as qualitative research methods are more and more replacing quantitative ones in the realm of what Watzlawick (1984) calls "second order realities" or "[t]he aspect of reality in the framework of which meaning, significance, and value are attributed" (pp. 237–238), there is a growing concern about *how* to handle the now "politically correct" issue of subjectivity of the researcher:

> Many authors currently focus on how subjectivity plays out in the actual conduct of research. Just as scholars advance different critiques of objectivity on an abstract level, they do not agree on how to respond to subjectivity in the practical conduct of research. They offer varying definitions of subjectivity. Some see subjectivity as taking sides and reject the idea of value neutrality (Boros, 1988; Roman and Apple, 1990); most accept that the emotions and predispositions of researchers influence the research process (Agar, 1980; Krieger, 1985; LeCompte, 1987; Peshkin, 1985, 1988; Rubin, 1981; M. L. Smith, 1980; Stake, 1981) and either term subjectivity as bias (Agar, 1980; Ginsberg and Matthews, n.d.; LeCompte, 1987), a quality of the researcher to capitalize on to enhance understanding (Krieger, 1985; Peshkin, 1985, 1988; Rubin, 1981; Smith, 1980), or interactivity (Eisner, 1990; Guba, 1990a). (Jensen & Peshkin, 1992, p. 703)

As far as we are concerned, subjectivity is not an either-or, but rather a both-and proposition: Yes, we reject the idea of value neutrality; yes, the emotions and predispositions of researchers influence the research process; yes, subjectivity is bias; yes, subjectivity is a quality of the researcher to capitalize on to enhance understanding; and yes, subjectivity means interactivity.

For us the real issue is which aspect of subjectivity to highlight in which context and for what purpose.

The focus of our linguistic action inquiry methodology has been one main aspect of subjectivity: *enhancing the person of the teacher/investigator as the main teaching/research instrument.* Utilizing herself as her main teaching/research instrument, the teacher/investigator is able to capitalize on her subjectivity so that both she and the students gain a deeper understanding of students' experiences. To put it in other words, as an exquisite teaching/research instrument, the teacher/investigator is able to successfully *separate* her own meanings from those of the students, and thus successfully guide them in the co-construction of new mathematics knowledge.

2.3 The person of the teacher/investigator as our primary teaching/research tool

Lesh and Lovitts (1999) ask the following question: "What knowledge and abilities must teachers develop when it is no longer possible to be an 'expert' in every area of student inquiry and when teachers' roles must shift from delivering facts and demonstrating skills toward being professional knowledge guides, information specialists, and facilitators of inquiry?" (p. 65). To answer this question, we turn to Lincoln and Guba (1985), who believe that effective inquiry requires congruence between the paradigm, model, the relationship with the evaluand, the framing of the problem, and the overall context. Often, in the context of training teachers, the very training methods themselves presuppose different epistemological assumptions than those we intend to impart. First, we devote most of our attention to imparting models and ideas, paying virtually no attention to our primary training tool: *the person of the teacher*. We believe, however, that a major objective in training teachers needs to be the enhancement of their persons as the teaching/research instrument, particularly in their role as "facilitators of inquiry."

We make the assumption that tacit awareness and thus the ability to become more congruent, may be enhanced by learning. Although it is often assumed that tacit knowledge is innate, we believe that intuition has structure and is teachable and learnable. One way a teacher might increase his or her tacit awareness is to model persons who are successful at incorporating students' behavior and perceptions, current and past relationships, existing life experiences, innate and learned skills and abilities into the teaching process (much like constructivist therapy models the work of Milton Erickson or Virginia Satir – see [Hale-Haniff & Pasztor, 1999]). One way to approach this might be to notice what they notice, attend to how they make sense of what they notice, and be able to respond as they respond. **Table 2**, outlined in an information processing format (Hale-Haniff, 1989), presents particular skills we deem essential for such enhancement of the teacher/investigator as a teaching/research instrument and that we use in our teaching/research in order to enhance our tacit knowledge.

We acknowledge that *acquiring* the skills we are describing in Table 2 is a distinctly different process from actually *using* them. *Learning* each skill involves conscious repetition of listening, observing, and performing the skill to a point that it becomes a fixed and unconsciously automated pattern (Hale-Haniff, 1989). Later, when the teacher is actually *doing* teaching, skills are accessed "naturally" as a function of unconscious pattern recognition. This type of learning has previously been described by M. C. Bateson (1972).

Sensory Input:	Processing :	Behavioral Output:
What you notice: the direction & flow of attention.	How you interpret and assign meaning to communication	What you say, how you say it, and body language
Ability to: • Learn to use all senses in more flexible, integrative ways • Strengthen acuity of all senses; not just the strongest sense system; accommodate to student's system of choice • Enhance sensitivity of calibrating responses of student and self • Detect instantaneous feedback from self and student regarding ongoing communication *fit* • Attend to non-verbal & verbal, process & content of communication • Detect patterns of congruence and incongruence	**Ability to:** • Be flexible in assigning multiple meanings • Infer meaning based on student's feedback rather than assigning own interpretation • Distinguish between what is sensed tacitly and what is an association to one's own past experience • Recognize and clarify ambiguous, abstract, and multi-level communication • Use the student's own norms or standards as basis for comparison • Maintain a flow state/awareness of wholeness	**Ability to:** • Demonstrate an even flow of attention until the student or teacher recognizes / punctuates something as important • Increase behavioral flexibility in what you say, how you say it, and body language • Translate communications to accommodate student's system • Verbalize own assumptions using tentative language and inflection • Respond to incongruent communications in ways that restore communication flow

Table 2. Areas for Enhancing the Person of the Teacher/Researcher (Adapted from Hale-Haniff, 1989)

In what follows we will highlight some of the major ways in which we actually implement the enhancement of the person of the teacher/researcher as the main teaching/research tool in our projects.

2.3.1 Attending to all aspects of sensory experience, including emotions

Positivist methodology privileges auditory-verbal communication, often to the exclusion of other modalities. In contrast, the holistic, constructivist view presupposes that the teacher/investigator should have the potential to attend to all aspects of sensory experience and communication *both* in herself and in the student's system. In addition to auditory-verbal aspects, visual and kinesthetic experience may also be privileged, with both unconscious (tacit) and conscious communication and perception considered.

In the positivist view, which tends to fragment human experience and emphasize the rational aspects, *emotions* have generally been conceptualized as separate and apart from the rest of human subjective experience. Most of us have been socialized largely according to positivist thinking, and may tend to think of emotions as sudden and intense experiences that come and go at certain times; something that a sane or balanced person learns to keep under control so that rational thinking and control can prevail. On the other hand, the holistic, constructivist view depicts emotional experience as ongoing, simultaneous with and supportive of the rest of experience.

Defined as changes in body states, emotions occur in concert with other mind-body experience—they are ever-present and manifest themselves in our minds in form of so called "body images": "By dint of juxtaposition, body images give to other images a quality of goodness or badness, of pleasure or pain. I see feelings as having a truly privileged status. ... [F]eelings have a say on how the rest of the brain and cognition go about their business" (Damasio, 1994, pp.159-160). "Body images" are of two kinds: "feelings of emotion" and "background feelings," the latter corresponding to our "body states prevailing between emotions" and contributing to our moods, to our proprioception, introception (visceral sense)—in general to our "sense of being" (Damasio, 1994, p. 150). It is important to note that experience that is kinesthetic to one person (say, a student) is accessible primarily visually to an observer (say, a teacher/inquirer). For example, as a student feels his or her face get hot, the teacher/inquirer might notice him or her blush. Or, as a student feels a sense of pride welling up in him, the teacher might notice him taking a deep breath as he squares his shoulders. Thus, learning to detect new categories of sensory experiences in ourselves and others involves enhancing perception of new categories of both kinesthetic and visual experience. By becoming more consciously aware of categories of sensory experience other than auditory-verbal, we enhance our ability to accommodate to the students' experiences.

2.3.2 Attending to physiological and language cues

Paying attention to sensory experience involves attending to people's distribution of attention across visual, auditory, and kinesthetic aspects of experience. Although sensory experience is simultaneously available to all senses, people attend to various aspects of see-hear-feel experience at different times. For example, let us take the case of two children trying to work together on a mathematics problem. One child does "not *see*" what they are supposed to do, while the other states she doesn't get "a *feel*" for what they are supposed to do. In this scenario, communication flow is obstructed because each child is attending to a different sense system, or logical level of experience (Bateson, 1972). By noticing this, we help the children translate their experience so it can be shared and attention can again flow freely. By paying attention to sensory experiences and their physiological expression, we help avoid sensory system mismatches that often take place between teachers and children. For example, if a child says, "Your explanation is somewhat *foggy*," the teacher's response of matching the visual system by asking "What would it take to make it *clearer*?" might be a better *fit* than the kinesthetic mismatch of "So you *feel* confused?"

People's sensory strategies (see section 3.1 herein for a definition) are processes that cause "changes in body state—those in skin color, body posture, and facial expression, for instance—[which] are actually perceptible to and external observer." (Damasio 1994, p.139). These physical reactions are important cues for the *external observation* and *confirmation* of people's sensory strategies. The primary behavioral elements involved are: *language patterns, body posture, accessing cues, gestures, and eye movements* (Dilts, Dilts, & Epstein, 1991; Pasztor, 1998; Hale-Haniff & Pasztor, 1999).

We attend to people’s *language patterns* based on the assumption, derived from constructivist therapy case studies and literature, that sensory experience or “the report of the senses” reflects the interaction between body and mind, and that one can attend to communication behavior as a simultaneous manifestation of sensory experience (Satir, 1967; Hale-Haniff & Pasztor, 1999). We pay special attention to metaphors people use in their language. According to (Lakoff &Johnson, 1999), a metaphorical idiom is “the linguistic expression of an image plus knowledge about the image plus one or more metaphorical mappings” (p. 69), and so it can also serve as a source of information about people’s sensory experiences. In the traditional, positivist view of metaphor, “metaphor is a matter of words, not thought”; it “occurs when a word is applied not to what it normally designates, but to something else”; metaphorical language “is not part of ordinary conventional language but instead is novel and typically arises in poetry, rhetorical attempts at persuasion, and scientific discovery”; metaphorical language “is deviant; in metaphor, words are not used in their proper senses”; and metaphors “express similarities, that is, there are preexisting similarities between what words normally designate and what they designate when they are used metaphorically.” The most widespread traditional view is that conventional “metaphorical expressions in ordinary everyday language are ‘dead metaphors,’ that is, expressions that once were metaphorical but have become frozen into literal expressions” (Lakoff & Johnson, 1999, p. 119). This traditional view of metaphor “has fostered a number of empirically false beliefs about metaphor that have become so deeply entrenched that they have been taken as necessary truths, just as the traditional theory has been taken as definitional” (Lakoff & Johnson, 1999, p. 119). The success of constructivist therapies in using linguistic metaphors as expression of people’s sensory experiences belies each of the traditional views. Below, we give some examples of linguistic metaphors that students often use in the classroom and that we utilize to calibrate their sensory experiences (Bandler & MacDonald, 1988; Lakoff &Johnson, 1999; Hale-Haniff, 1989). They are categorized according to the primary sense system they presuppose.

VISUAL: I see what you mean. That’s a murky argument. Things were blown out of proportion. Shrink the problem down to size. You are making this bigger than it is. It is of small importance. The problem is larger than life. It is a big problem. It is of minuscule importance. It is a major issue. It is of peripheral importance. I need to see it from a new angle. I don’t see the big picture. This is a new point of view. Let us look at the other side. The problem towers over me. The solution was in front of my nose. This problem seems overwhelming. I need some distance from it. That throws a little more light on it. It all seems so hazy. I don’t know—it just flashed on me. When you said that I just saw red. Well, when you frame it that way, yes. I need to bring things more into perspective. Everything keeps spinning around and I can’t seem to focus on one thing. It’s too vague even to consider. It’s off in the left field somewhere. The image is etched in my memory. I just can’t see myself being able to do that. I’m moving in the right direction. I can’t face it. It’s not a black and white world. This is top priority. Let’s look at the big picture.

AUDITORY: It rings a bell. It sounds right/familiar. The right decision was screaming at me. She gives me too much static. It’s just a whisper. If I nag myself long enough, I’ll do it. Got you, loud and clear. We need to orchestrate our solution. It came to a screeching halt. I keep telling myself, “You can’t do anything right.” It’s too off-beat. He tuned in. This is an unheard of solution. It has a nice ring to it. He talks in circles.

KINESTHETIC: I cannot grasp it. It feels right. The solution hit me. This is hot stuff. Whenever I hear that, my stomach knots up. The pressure is off. The whole thing weighed on my mind. I’m off center, like everything is out of kilter. I’m trying to balance one against the other. Yeah, I feel up to it. It’s an esthetic solution. It all boils down to this. It slipped my mind. It’s a perfect fit. He brushed it off. I am tossing ideas around. Get in touch with my intuition.

Acccording to Lakoff and Johnson (1999), except for an inherent, literal, nonmetaphorical skeleton, all abstract (and hence also mathematical) concepts are built on primary metaphors.

“Correlations in our everyday experience” on the other hand, “inevitably lead us to acquire primary metaphor, which link our subjective experiences and judgments to our sensorimotor experience. These primary metaphors supply the logic, the imagery, and the qualitative feel of sensorimotor experience to abstract concepts. We all acquire these metaphorical modes of thought automatically and unconsciously and have no choice as to whether to use them” (Lakoff & Johnson , 1999, p. 128). Our sensorimotor experience is expressed not only through language, but through all of our behavior. For example, “when we gesture spontaneously, we trace images from the source domain in discussing the target domain …” (Lakoff & Johnson , 1999, p. 127). By carefully attending to communication behavior cues in an ordered manner, in her therapeutic practice Satir was able to help her clients co-construct desired experiences. These behavioral cues fit into the general categories of *what you say, how you say it, and body language* (Satir, 1967). **Table 3** summarizes examples of communication behaviors by the sensory modality they presuppose.

		Visual	**Auditory**	**Kinesthetic**
What you say		visual predicates	auditory predicates	kinesthetic predicates
How you say it		higher pitch; less variety of inflection; quality may be nasal or strained; higher rate of speech	mid-range pitch; varied & melodic inflection; moderate, rhythmic rate	lower pitch; longer pauses, breathy slower rate *or* higher pitch; few pauses, shrill, faster rate
Body Language	**Breathing**	breathing high in chest, shallow, and more rapid	breathing mid-chest; and moderate rate	breathing low in abdomen, slower rate or holding breath *or* whole body heaving with breath; & exaggerated rate
	Eyes	may squint or defocus; eyes may converge to a given point in space, upward eye movements	side to side eye movements	may lower eyes
	Arm, hands, & fingers	gestures toward eyes; upward movements of arms; may gesture to particular spatial locations	gestures around ears and mouth; may cross arms; snap fingers; place hand on chin (telephone position)	gestures toward lower abdomen, mid-line of torso or heart; hand gestures with palm facing body; fingers may move in sync with rhythm of body sensations; Arm and hand gestures may trace sequences of body sensations.

Table 3. Three Levels of Behavioral Cues for Identifying See-Hear-Feel Strategies (Hale-Haniff, 1986)

Awareness of behavioral cues has the benefit of dispelling misconceptions that parents and teachers often have about children’s behavior. You have probably heard parents or teachers say to their children, “The answer is not on the ceiling!” while forcing them to look down on their notebooks when doing their homework or taking a test. In doing so they inadvertently keep the children from accessing information visually and instead lock them into the kinesthetic modality.

This is of particular significance in mathematics, where visualization is often the key to solving a problem (Wheatley, 1997; Presmeg, 1997). You have probably also heard parents or teachers say to their children, “Look at me when I talk to you!” When people listen, they have a natural tendency to turn an ear toward the sound source, so facing it will not come naturally to them. Sometimes we force our children to look at us while we talk, and then we complain that “you haven’t heard a word of what I said, have you?” You have also probably heard parents or teachers say to their children, “Stand still when I talk to you!” While we don’t have much room here to discuss body movement, we want to emphasize that being able to recognize its correlation to internal processing might be a critical tool for helping someone access optimal learning states. It may also be all it takes to categorize a child as “gifted,” as opposed to “at risk.”

Sociologist Lilian Rubin (1981) talks about the importance of clinical training as a tool that “helped her to establish rapport, to detect ambivalence, and to give importance to what is said and not said” (Jensen & Peshkin, 1992, pp. 708-710). Valle uses her clinical training to help students access resource states of learning. She remembers: “I had this student, and after 5 minutes of sitting in his chair, he got real antsy. So I worked with him on finding something to help him get back in the classroom. His hands got real hot, and so I would come by and all I had to do is touch his shoulder and he would know to grab hold of the legs of the chair because they were steel and they were cold and you could just tell he got a relief. Another student that I have who has difficulty staying on task for long periods of times—I time him and after ten minutes I notice him going off. All I have to do is have him do this [kinesiology] exercise where they get up on their toes and they run their eyes along the line where the walls connect up and to one side and find which side is more comfortable for them while keeping their heads still. Then they sit down and are able to work for 20 minutes.”

If a person is using gross body movements—large motor movements compared to fine motor movements—we instinctively know what the relationship between the *level of detail and the level of abstraction* (in the submodalities—see next subsection) of his or her internal processing is. It would be really odd for that person to say “I got the details, now give me the big picture.” The more precise the body language, the more precise the “chunk size” of information. We can also tell the high degree of detail by the narrowing of the gaze—it’s almost as if the person was focusing on a particular area of the fine print as opposed on a diffused thing, such as noticing a page or a computer screen. Duration and intensity of gaze, coordination of eye and head movements, head tilt and angle, chin orientation (up, down and middle)—some of these are *accessing cues*. They might tell us the state that people are in, the configuration of their attention, level of detail, what they are attending to. Sometimes people lean their head to one side when they are receiving new information, and to another side when it is “a rerun.” Noticing these cues can be very helpful to see that a student is receptive to what we are saying or when his system is closing down a bit. In the latter case, how can we shift the way we are presenting information so that he opens back up again?

Let us say, for example, that a student wanted to learn a subject area and we noticed his physiology starting to shut out new information. Then we map the precise point where he shut down and figure out what was going on that caused him to shut down, in order to help him get back in state. (We also are careful not to comment about what we are just doing, because otherwise students start to feel uncomfortable. If we shift a person into self-consciousness, we break the very state that we are trying to elicit.)

2.3.3 SEL revisited: Submodalities—refining the see/hear/feel components

When we attend to physiological and language cues of students’ experiences, we attend to much more than just which sensory modalities they are using and when. Each sensory modality is designed to ‘perceive’ certain basic qualities called *submodalities*, of the experience it represents (Bandler & MacDonald, 1988; Pasztor, 1998; Hale-Haniff & Pasztor, 1999). The premise of our research is that submodality distinctions are not there to be discovered, but are co-constructed in the process of communication. Moreover, the teacher/investigator *embodies* these distinctions in her neurology and mindfully reflects them in her language and communication with the students, thereby creating a basis for a shared experiential language and she is able to literally “make more sense” of her students.

Table 4 lists some submodalities for each sensory modality together with the kinds of questions we ask in order to facilitate their co-construction (adapted from Bandler & MacDonald, 1988).

Sensory modality	Submodality	Eliciting question
Visual	Location in space	Show me with both hands where you see the image? More to the left, center or right? (May also gesture with eyes or describe verbally.)
	Distance	How far away is the image? By the door? Across the street? Three feet away?
	Relative size	How big is the picture compared to life -size?
	Color/black and white	Is it in color or black and white? Are there a lot of colors? Are the colors real bright or are they washed out?
	Degree of clarity or focus	Does the picture seem sharp and focused or is it fuzzy?
	Movement within the image	Is it a movie or a still picture? How fast is it going compared to normal?
	Movement of the image	Is the image stopped in one place? Which way does it go?
	Detail	Do some things seem closer and other things farther away? Is it easy to see the tiny detailed parts of the whole picture, or do have to make a special effort to see them?
	Brightness	Is the lighting brighter or darker than normal?
	Orientation	Is the picture straight, or is it tilted?

Sensory modality	Submodality	Eliciting question
	Associated/dissociated	Do you see the events as if you were there or do you see yourself in the picture?
	Border	Is there a frame around it or do the edges fuzz out?
Auditory	Content	Is it voice, music, or noise?
	Location	Do you hear it inside your head or outside? Where does the sound come from"?
	Pitch	Is it high-pitched or low-pitched? Is the pitch higher or lower than normal?
	Volume	How loud is it?
	Tempo	Is it fast or slow?
	Rhythm	Show me the beat or rhythm.
	Duration	Does it stop and start, or does it go on and on?
	Mono/stereo	Do you hear it on one side, both sides, or is the sound all around you?
Kinesthetic	Location	Show me where you feel it in your body? Where does it start and where does it move to? How does it get from the place it starts to the place where you feel it the most.
	Movement	Does the feeling change as it moves? Is it moving all the time or does it come in waves?
	Speed	Show me how it moves. Does it move slowly or quickly?
	Quality	How would you describe the body sensation: tingly, warm, cold, relaxed, tense, knotted, sharp, spread out?
	Intensity	How strong is the feeling?

Table 4: Submodality distinctions and questions to facilitate their co-construction (adapted from Bandler & MacDonald, 1988).

Submodalities are distinctions that separate experiences from each other. As such, their significance comes to bear only when we contrast submodalities of images that come from different experiences. To illustrate this, let us look at the submodalities of different experiences of

Michael, specifically at how different contexts are manifested in completely different sets of submodalities. Michael is an architect and he is quite proficient in geometry. First, here is what he reports regarding his (mostly visual) experience of abstraction: “As part of a math problem involving triangles, an *abstract* triangle occurs first as a fuzzy shape without any material ‘body.’ It doesn’t have a surface, not even a clear boundary. Its size is also changing between a couple of inches to one or two feet. It is quite far from my face and its distance is unspecific but it is still in the room. As a consequence, its shape, size, and location can easily be manipulated. As it is manipulated, like made equilateral or rotated, these parameters change rapidly. The boundary becomes more defined, the size concrete, and the distance fixed. It still remains, however, a line-drawing without a body or surface. It is always a colorless figure either gray or black and white. There is no definite feeling attached to the pictures. However, the more abstract the picture, the further it is removed from any emotion.”

In contrast, imagining an emergency triangle on the road “propped up behind a car is a vivid picture with concrete shape, thickness, material, and so on. It is red with white edges in fluorescent colors set against the gray asphalt background. I see it at a distance of 10 feet in life size, that is, the same size I would probably see it driving by and looking at it from this same distance. I feel some anxiety in my stomach as I probably connect this picture unconsciously with a car break-down or an accident.”

As yet a third example of submodality experience, Michael turns to his experience solving complex geometric problems. First, he describes the context: “Basic properties of the conic sections can be proved visually by using the so-called Dandelin spheres (named after G.P. Dandelin, 1794-1847, who was a French engineer and lived in Belgium). The proofs of these properties are based on intersecting an infinite double-cone surface by a plane. Depending on the relative position of the plane to the cone, the section is a circle, an ellipse, a parabola, or a hyperbole (not including the extreme case when the plane intersects the cone exactly through its tip). In each case we can inscribe two spheres into the cone, each touching the surface of the cone in a circle and the plane in one point. This latter point is the focus (or one of the foci) of the conic section. The properties of the conic sections are proved by reasoning about the relation between the distances between a point of the conic section and the foci to the circles in which the spheres touch the cone surface. This relation can be easily seen on a sketch or a mental picture.” Here’s Michael’s “mental picture”:
“The picture of the cone, the plane, and the two spheres are initially abstract without much physical properties. This is due to the fact that the cone and the plane are infinite, but also I don’t want to specify initially the opening of the cone and the sloping of the plane. As I proceed to narrow down the task, the picture becomes more specific. I choose a specific opening for the cone and a certain, let’s say, 30 degree angle for the plane. The result is that the upper part of the cone disappears as it becomes irrelevant. The picture moves closer and becomes bigger but remains abstract. It still has fuzzy boundaries and no material qualities. The picture takes on a 3D line-drawing quality as I try to imagine the curve, in this case an ellipse. As I focus on details such as imagining how the sphere touches the plane, I zoom in further and neglect the rest of the picture. The part I zoom in on becomes clear and obtains some material quality, like a paper model. However, it is transparent. It is just in front of me about two feet away. I could touch it and manipulate it with my hands if I wanted to. When I move to other details, like the bottom sphere or the two touching circles, then I move the picture back into the previous position and zoom in on the new details. The picture never has color, solid shape, material features (like wood or glass), or any movement other than my intentional moving it back and forth for manipulation. There are no feelings attached to these pictures.”

A nice expression of modalities/submodalities at work comes from a time in our pilot project when we presented the children with the following problem taken from (Wheatley, 1997, p. 289):

“Imagine a five by five by five cube [made of unit cubes]. Paint is poured down over the top and the four sides. How many [unit] cubes would have paint on them?” One of the children worried that we might need to use some other, thinner substance to pour over the cube, as paint may be too thick and may not cover the cube evenly. Other children immediately asked whether paint could get underneath the cube or into the cracks between the unit cubes. (Remember, there was no actual physical cube or liquid present at this discussion.)

2.3.4 Attending to Process v/s Content

While attending to the children, we, as teachers/investigators, pay attention to the communication *process*, not just the *content.* While content generally refers to *what* is talked about, or *why* it is talked about, process refers to the *how* of the way problems and solutions are communicated. Process, or pattern-based distinctions occur at different logical levels of communication than content-based distinctions do (Bateson, 1972). Attending only to content makes it far more likely that the teacher/investigator will associate elements of the student’s communications with his or her own private meanings rather than with the student’s. Also, by attending to process rather than only to content, the teacher/investigator can detect order or pattern, using *other ways of knowing* besides rational logic such as we have described in previous sections.

2.3.5 Rapport

It is of utter importance to note that we don’t and will never use the language and physiological patterns we observe without always first comparing the information to something to contextualize and give it meaning. For example, changes in posture, physiology, or affect are detected by first having calibrated the student’s overall attitude or stance during a particular class.

We use this information in various ways, depending on the intended outcome of the classes. However, regardless of outcome, we always feed back information calibrated to the students to test for accuracy and recognition. If our hypotheses are not accepted by the students, we revise them and we recalibrate communication. A very important tool hereby is the building of *rapport*. In the constructivist view of communication, rapport is an ongoing, moment-by-moment process of developing and maintaining communicative *fit* with the student’s representational system. “*If* a teacher is able to recognize the representations that a student is using, and can make contact with these representations, the resulting discussion is nearly certain to be helpful to the student (see e.g. Maher, Davis, & Alston, 1992). … When a teacher *fails* to recognize the representations that a student is using, the ensuing discussion can be disastrous, as in one of our videotaped classroom episodes … (Davis &Maher, 1990)” (Davis & Maher, 1997, p. 98).

Conceptually, the goal is to set a joint intention with the children and continually accommodate ourselves to what we think is going on in the children’s head and body, thus maintaining a communicative *fit* between the children and ourselves; each is predisposed to notice and approximate aspects of each other’s behavior in line with our joint intention.

Finding joint intentions with the students is fundamental to constructivist education, but is a topic beyond the scope of this paper. It concerns the process of achieving or maintaining congruence with self and others, the relationship among the systems of values, goal setting and intention, attention, emotion, and behavior. It is also a concern that has been addressed by NCTM (2000): Often “the curriculum offered to students does not engage them. Sometimes students lack a commitment to learning” (chapter 1), which, of course, goes hand in hand with a lack of motivation to study mathematics. For now, we refer to (Hale-Haniff & Pasztor, 1999) (see Diagram **I** therein).

2.4 Situating our Methodology in Qualitative Inquiry

Qualitative research is characterized by use of multidisciplinary approaches, multiple methodologies, and utilization of the person of the researcher as the primary research instrument. For inquiry to be effective, there must be congruence across the choice of paradigm, methodology, the relationship between co-participants, situational objectives, and the overall context of mathematics (Crabtree & Miller, 1992; Lincoln & Guba, 1985; Schwartz & Ogilvy, 1979). Our Linguistic Action Inquiry Project and thus our pilot project employ participative inquiry as their methodology. Participative methodologies include cooperative inquiry, participative action research, as well as action science and action inquiry (Reason, 1998). Our projects in particular, employ action inquiry.

Action inquiry, developed by Torbert (1981), builds on action science (Argyris & Schön,1974, 1978; Argyris, Putnam, & Smith, 1985; Schön, 1983). Action science involves setting an intention, taking action based on that intention, factoring in feedback based on that action, and then taking further action. Action inquiry (Torbert, 1981) builds on action science by paying special attention to the importance of "outcomes … and the quality of one's own attention monitored by meditative exercises as one acts" (Reason, 1998, p. 274). For Torbert (1981), the "primary medium of research is an attention capable of interpenetrating, of vivifying, and of apprehending simultaneously its own ongoing dynamics and the ongoing theorizing, sensing, and external event-utilizing (Torbert, 1972). Only such an attention encompasses purposes, strategies, actions, and effects. Thus, only such attention makes it possible to judge whether effects are congruent with purposes—i.e. whether an acting system is effective" (Torbert, 1981. p. 148).

According to Torbert (1976) (cited in Reason, 1998), developing this quality of attention requires rigorous discipline and an "unimaginable scale of self development." In exploring the issue of personal development, Torbert draws on "the ancient tradition of search for an integrative quality of awareness and on modern theories of ego development" (Reason, 1998, p. 275). In a similar vein, Reason (1998) proposes bracketing off our own discourse as researchers and thus approaching experience more directly through "mindfulness disciplines (meditation, T'ai Chi, Gurdjief work, Alexander Technique), through consciousness-raising, and through systematic engagement with the cycles of action and reflection that are a central part of participative and action inquiry methods" (Reason, 1998, p. 281).

For Torbert (1991), "action inquiry is 'a kind of scientific inquiry that is conducted in everyday life.' Action inquiry differs from orthodox science in that it is concerned with 'primary' data encountered 'on-line' and 'in the midst of perception and action' and only secondarily with recorded information. Action inquiry is 'consciousness in the midst of action' (p. 221)" (Reason, 1998, p. 275).

What we have added to the method of action inquiry is a set of sensory-based distinctions which constitute categories of a language that can be spoken from "the midst of perception and action." This language differs from more usual language use in that it is not used to talk *about* an experience: the languaging *is* the experience. As a multidisciplinary team we believe that some of the founders of family therapy (Virginia Satir and Milton Erickson) intuitively and independently embodied what Torbert discusses in his ideas on developing the person of the researcher as the primary research tool. By organizing the sensory-based distinctions presupposed in how these masters used their attention, we have developed a language (SEL) that reflects these categories. Parts of this language have been variously described in a number of fields of which we are aware of, such as Damasio (1994) in neuropsychology, Csikszentmihalyi (1978; 1990; 1997) in social psychology, and Satir (1967) and Erickson (1958) in family therapy. We have abducted and synthesized their writings and works, hereby creating the origins of our shared experiential language based action inquiry methodology, coupled with the new, embodied linguistics work of authors such as Lakoff and Johnson (1999).

3. The Pilot Project

The pilot project is an application of our linguistic action inquiry methodology to the mathematics classroom practice. It employs linguistic action inquiry in two ways: longitudinally in cycles of planning, doing, checking and acting, but also situated in a given context of mathematics learning and the context of moment to moment communication using SEL—the shared language of intersubjective experience, described earlier in part 2.

The goal of the pilot project is twofold:

1. to investigate how the methodology of linguistic action inquiry can help successfully root mathematical understanding in students' prior sensory experiences, and
2. to learn, utilizing SEL, how students naturally organize their experiences when they try to understand mathematics.

In contrast to traditional, formal academic educational research which has been employing a process consisting of setting a task, recording solutions, and analyzing childrens' solutions from an adult frame of reference, our project helps students reconnect their mathematics understanding to *lived experience* through participation—a "living processes of coming to know" (Reason, 1998, p. 263). And since "knowledge arises in and for action," the primary outcome of our project "is a change in the lived experience of those involved in the inquiry" (Reason, 1998, p. 279).

By the very nature of our research goal, we have to explore how consistent understanding plays out experientially for each individual student, and so our research advances through various action-research cycles, spiraling through phases of

- offering new knowledge (using The Math Advantage Daily Practice for the Florida Comprehensive Assessment Test [FCAT] for the fourth grade, published by Harcourt Brace, as our main source of mathematics problems, mainly to make sure that the project classes do not in any way disrupt the students' regular mathematics instruction, in which—unfortunately—the focus is preparation for the FCAT),
- eliciting from the students the experiences the knew knowledge triggers,
- testing it for *fit* with the purpose of consistent understanding,
- adjusting *fit*,
- backtracking to or creating more useful experiences as necessary,
- and reflecting on the next course of action.

While our methodology gives our research a direction, we never know what the outcome of any of the action phases is going to be. The outcomes, however, lead to new insights and knowledge that inform our next action. By choosing linguistic action inquiry as our methodology, we are able "to ground knowing and action literally in the body of experience—'coming to our senses,' as Berman (1989) puts it" (Reason, 1998, p. 282).

Our *data collection* techniques focus on modeling the students' mathematics understanding experiences triggered by the various tasks at hand. We collect students' homework and in-class writings (of which we give some samples in this and the next section), and Valle or at times a teacher's aid, takes notes during classes. Besides such data, our methodology also includes as data "a whole range of personal experience and idiosyncratic expressions, and although primarily verbal, reaches toward what he [Torbert] terms the 'meditatively postverbal'" (Reason, 1998, p. 282). These data serve as our main source to inform us in our next course of action to adjust *fit* between students' existing experiences and the mathematical knowledge we are trying to help them construct. To illustrate this, let us look at a particular episode that occurred in the beginning of the project. Exploring their experiences with numbers and the importance of the number "1" in particular, Pasztor asked the students, "What would happen if we only had the digit '1' and no other digit. Could we live with that?" The responses took her by surprise:

Allene[2]: "We couldn't count, we also couldn't have 2 or more of any thing except 11 or 111 etc."

Ron: "If we did not have the other digits we would not be able to make any food because if it needed 2 of something and it really needed 2 of that ingridient you could not make it."

Les: "If someone asked how old you are you'll always be 1 year old."

Sally: "Without the other digits there would be one of everything. So if someone wanted the same thing that someone wanted there would only be one of that thing."

Mike: "If we have only number one, there would only be one of everything. And also one person. And he would eventually dies."

Simone: "We cannot live with not having the other digits because then I wouldn't be nine right now."

Karl: "Everybody could have 1 kid because they can't count higher also we'd die quickly because we would only live 1 year."

Some children did understand that using the digit "1" we can denote any other number—like for example, three by "1+1+1" (but they agreed that "it wouldn't be fun"). Most children, though, Pasztor realized, were confusing numbers in their role as denotation or symbols with the concepts they denoted or symbolized. This prompted her to design some new activities that would help the students understand what was happening, including thinking about names and about whether a person who didn't have a name didn't exist.

The *population* of our pilot project is a class of fourth graders in the "gifted" program in a public school. There are 24 children in the class, 15 boys and 9 girls. Their ages range from 9 to 10. They come from lower-to-middle socioeconomic homes. Of these children 18 are Hispanic, 4 Caucasian, and 2 Asian. So far we have met with the students twice a week during their resource classes over the course of two terms—more or less regularly, depending on their class schedule.

At the beginning of the project we introduced the children to the idea of becoming more aware of "what is going on in their minds and bodies" while doing mathematics. We introduced them to SEL as the language with which they can express and share these experiences in the context of mathematics using sensory modality and submodality distinctions. Pasztor told the students that in the project classes they are going to be the actual teachers teaching her how they experience problem solving in terms of the distinctions of SEL. So students see themselves as active participants of the project. We were amazed at how readily they learned to view their experiences through the new lenses of these distinctions. Because we share an experiential language with the children, we are able to use our common language to understand their *in vivo* presenting mathematical awareness, and then accommodate the teaching of new skills/concepts to their existing experience.

[2] Students' names are pseudonyms.

Already at an early stage in our pilot project we noticed that the children's awareness of their subjective experiences in terms of sensory-based distinctions brought about a shift in their problem solving strategies. The children reported an increased sense of control, a sense of "slowing down" the process of problem solving and having more choice. Collectively, they began to speak the newly acquired experiential language.

Trying to explore with the children multiple ways of performing arithmetic operations, at the beginning of the project Pasztor asked them to explain why they performed the operations the way they did. Overwhelmingly they responded that they performed the operations the way they did because that was how Mrs. K (their mathematics teacher) did it, and "she was taught that way, too." They trusted that she showed them the "easiest way to do it." Later in the project, however, the students tended to explain their ways to solve problems more in terms of their own sensory experiences.

In the next section we illustrate by some examples how we have been able to utilize our methodology to enhance children's mathematical understanding, as well as our own ability to communicate among ourselves in an explicit manner.

3.1 SEL revisited: Children's Sensory Strategies

So far we have defined the see/hear/feel components of SEL, our shared experiential language that is the basis of our methodology, together with their basic qualities called submodalities. What puts these elements together are our thought processes. These are organized in sequences of images that have become consolidated into functional units of behavior leading to a particular outcome and often executed below the threshold of consciousness. We will call these sequences *sensory strategies*. Each image triggers another image or a sequence of images. For example, you hear X's name, this triggers your remembering X's face, close up, somewhat distorted, and pinkish red, which, in turn, triggers a negative feeling. Over time, each image or sequence of images comes to serve as a stimulus that automatically triggers other portions of the perceptual or recalled experience it represents. The creation of such triggers happens through learning and depends on various complex subjective, social, cultural and other factors also captured in Searle's (1992) notion of the Background.

In the first phase of our pilot project, we introduced the children to the shared experiential language SEL and elicited their sensory strategies for solving math problems. Below are some examples. Our goal in this phase was twofold: 1. to use the process to help students embody SEL, and 2. to assess the *fit* of school expectations with the students' in vivo mathematics understanding experiences, and backtrack to a point where we could co-create a bridge to a better *fit*.

Ray[3] chose the following problem to solve: Which measure is the best estimate to describe the length of the salamander below (picture followed text). Circle the best estimate:

3 inches 3miles 3 pounds

Here is what Ray wrote: "What I did was picture a huge ruler in front of my face and I saw the numbers 1,2,3,4,5,… I looked at the picture [in the book] and compared it with 3 inch and it was right. Besides, pounds is weight and miles is larger than inch."

Karl's strategy for implementing a pattern is also quite remarkable. One time in the pilot project, Ana (Pasztor) asked the children to multiply 1x1 (=1), 11x11 (= 121), 111x111 (= 12321), and 1111x1111 (= 1234321). Then she asked them to continue the pattern. Karl reported the

[3] Students' names are pseudonyms.

following for 11111x11111: “First I looked, then [knocking with his left hand on his head right above his left ear] I heard ‘tap, tap tap, tap tap tap, tap tap tap tap, tap tap tap tap tap, and then back down tap tap tap tap, tap tap tap, tap tap, tap.” He followed this up by writing 123454321.

Mindy repeatedly demonstrated a distinct problem solving strategy that lets her know that the result (in this case “50”) “is right.” Let us look, for example, how she solved the following problem: Alana entered the county spelling bee. She spelled 47 words correctly before she made a mistake. If she had spelled three more words correctly, she would have spelled twice as many words as last year. How many words did she spell correctly last year? A. 25 B. 27 C. 32 D. 35

Here is how Mindy explained her solution (in terms of what she saw, heard or felt) in her homework: “I added each number to itself and 25+25=50. The problem says 47 then +3=50. I did not feel anything but in my head I saw 47+3=50. I also saw that 50 was really gold and yellow and it was blinking and heard it beep. Beep, beep, beep, beep it sounded really fast and loud. My head was here [smiley face] and the numbers were here [smiley face below the first smiley face, shifted to the right, suggesting that she saw them in front, somewhat to the side]. The numbers were that == big. The other numbers were black besides 50. The numbers were very clear. I saw the numbers for about a minute. I saw the numbers after the question. I saw the numbers in numbers not letters. The same thing happened with 25+25=50.”

Often we will be asked whether students self-reported sensory experiences are “real” or “right” or “true,” or do they just make them up? Our answer is that in a constructivist world-view experiences are not out there to be discovered or reported, but are co-constructed in the context of communication for a certain purpose, and the only criterion by which we judge them is their usefulness for that purpose.

Our goal is to utilize co-creation of subjective experience to help students learn. Paying attention to the process-based distinctions that are the basis of SEL, our shared experiential language, requires detailed attention to “differences that make a difference” for co-constructing new experience. Just like submodalities, all distinctions, such as behavioral cues or modalities, become more evident *by comparison*. An excellent example is Mindy’s strategy when she was not able figure out a solution. The problem she was tackling was the following: Joey has a new puppy. His sister, Jenna, has a big dog. Jenna’s dog weighs eight times as much as the puppy. Both pets together weigh 54 pounds. How much does Joey’s puppy weigh?

Here is Mindy’s report: “At first I subtracted 54-8. I got 46 but then I felt that it was wrong. So I made sure then I realized that both of the dogs together weighed 54 pounds so then I subtracted 8 to 46 and got 38. Then I guessed 36 so I added it and got 74 so then I subtracted 74-20=54 so I thought okay since 74-20=54, I have to take 10 away from 36 so I get 26 I added 28 and got 54 but I also felt that was wrong so I checked it and realized that 26 and 28 have only a 2 number difference and it had to be 8. So I added 8 to 28 and got 36 I added 36+28=54. I did all the math in my head so I saw it in my head and I heard myself saying all the math.” It is interesting to note that Mindy “feels” when she is wrong, but she “knows” when she is right. By the way, children loved and adopted right away Mindy’s strategy of seeing the result blink when she “knows” she is right.

As part of our methodology, we often ask the children to “try on” each other’s sensory strategies. By doing so, they all are by comparison able to gain more awareness of their own strategies. For example, one time Karl had to solve the following problem in class: How many sides are on seven hexagons? Karl described, demonstrating with his hands, that he saw the seven hexagons in front of his face, about a foot away, in turquoise, arranged in two groups—one group of four hexagons on the bottom, and another group of three hexagons on top. Then he heard a voice, loud and clear (he demonstrated the volume and pitch) coming in from both sides telling him “seven times six is forty-two.” Upon asking whose voice it was, he said it was his grandmother’s voice.

While hearing the voice, the numbers appeared big and red to the left of the group of three hexagons. The numbers disappeared as soon as the voice stopped talking. Karl gestured again to show how big the numbers were. He wrote 7x6=42 in the air, with his left hand, from left to right, a little above eye level.

We then asked the children to "try on" Karl's strategy and tell us how they experienced it. Ray reported that everything went fine until he came to grandma's voice. "I don't know his grandma's voice, so I just replaced it with mine and then I continued." Kay, on the other hand, was unable to hear anything, so she just did the "seeing part." It is interesting to note that Kay's successful strategies are overwhelmingly visual.

Like most children in our schools, the children in the pilot project proved to have most difficulties in solving word problems. Using our linguistic action inquiry methodology, we are able able to backtrack to a point in their experience where their difficulties started, thus being able to fine-tune teaching. For example, recently Pasztor designed a series of lessons using (Van de Walle, 1998, Chapter 7: Developing Meanings for the Operations) for the purpose of helping students build understanding of more complex word problems from problems that *fit* their (then) present experience. Here is a sample lesson for multiplication and division:

Below, you will find 3 word problems. Please solve them and *for each* of them

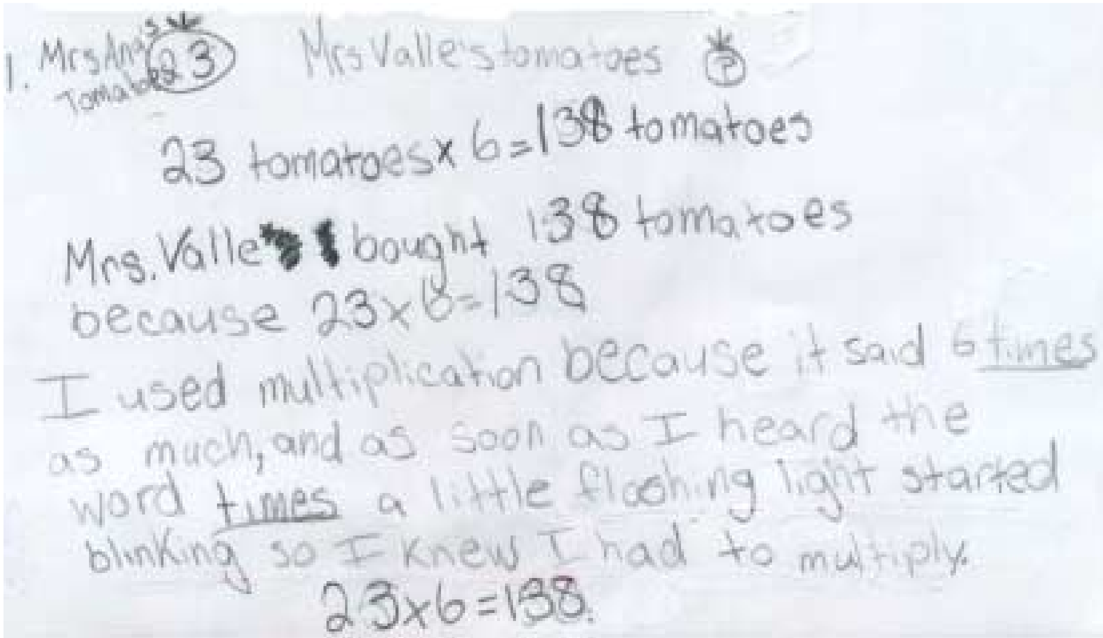
- draw a **picture** that shows what went on in your head while you were reading the problem and that goes with the solution,
- give a full-sentence answer to the problem,
- give an *explanation* for your solution, and
- give an *arithmetic equation* that goes with your solution.

Here are the problems.

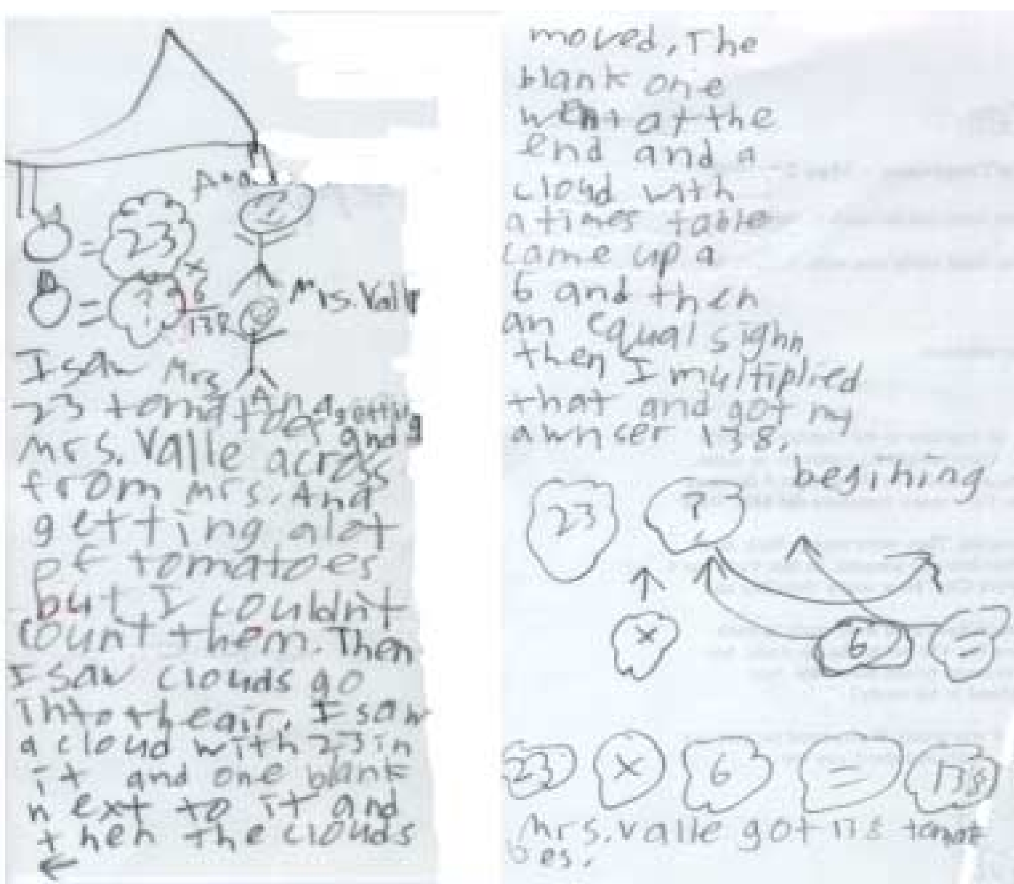
1. Mrs. Ana Banana/Tropicana and Mrs. Valle went together to the Sunday farmer's market in Coral Gables. Mrs. Ana Banana/Tropicana bought 23 tomatoes to make marinara sauce. Mrs. Valle's family loves marinara sauce, so she bought 6 times as many tomatoes as Mrs. Ana Banana/Tropicana. How many tomatoes did Mrs. Valle buy?
2. In Paris, France, there are 234 churches and temples. They were mostly built in the Middle Ages. Nowadays people build fewer churches and temples. In fact Paris has 6 times as many churches and temples as New York City. How many churches and temples does New York City have?
3. Mrs. Ana Banana/Tropicana and her husband each have their own study in their house. In her study, Mrs. Ana Banana/Tropicana has 456 books. In his study, her husband has only 76 books. How many times as many books does Mrs. Ana Banana/Tropicana have in her study as her husband in his study?

It is most remarkable how quickly, using SEL allowed the children to move from a syntactic problem solving strategy with an automatic understanding of what words go with what arithmetic operations to a semantic strategy where they utilized the whole context of the problem. Let us look at some examples. Please note how the use of SEL also reveals the children's reading strategies. A lot of children reported "hearing" the words, even though they read the problems—the problems were not read to them.

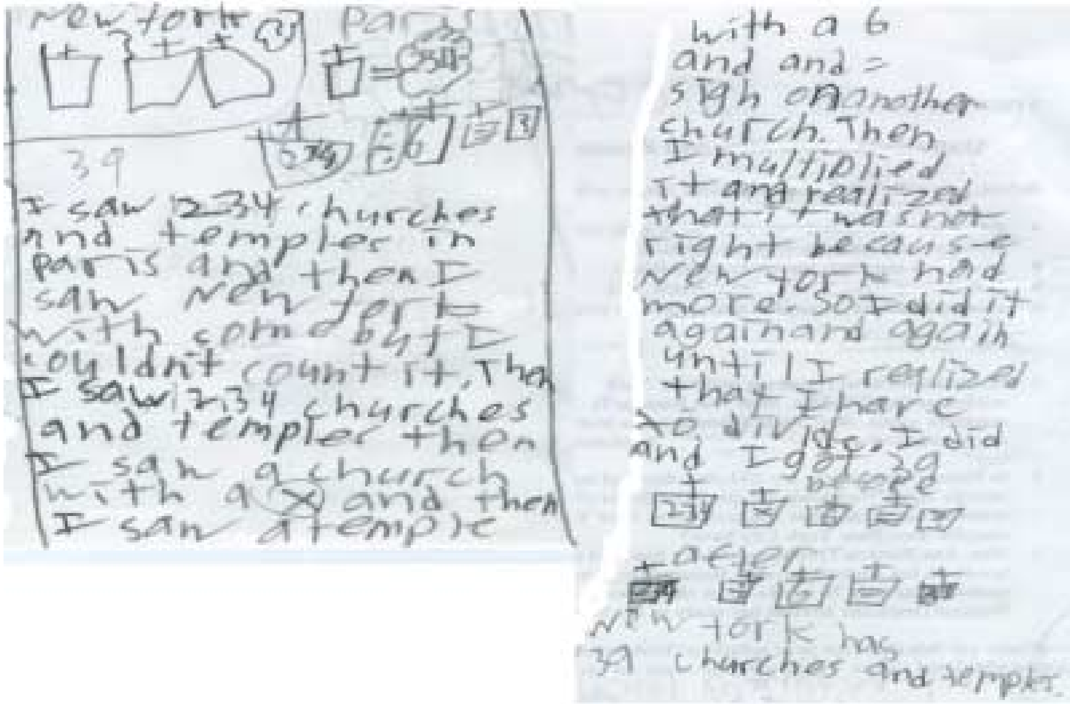
Kay, problem 1: "I used multiplication because it said 6 times as much, and as soon as I heard the word <u>times</u> a little flashing light started blinking so I knew I had to multiply."



Mindy, problem 1: “I saw Mrs. Ana getting 23 tomatoes and Mrs. Valle across from Mrs. Ana getting a lot of tomatoes but I couldn’t count them. Then I saw clouds go into the air. I saw a cloud with 23 in it and one blank next to it and the clouds moved. The blank one went at the end and a cloud with a times table came up a 6 and then an equal sign then I multiplied that and got my answer 138.”



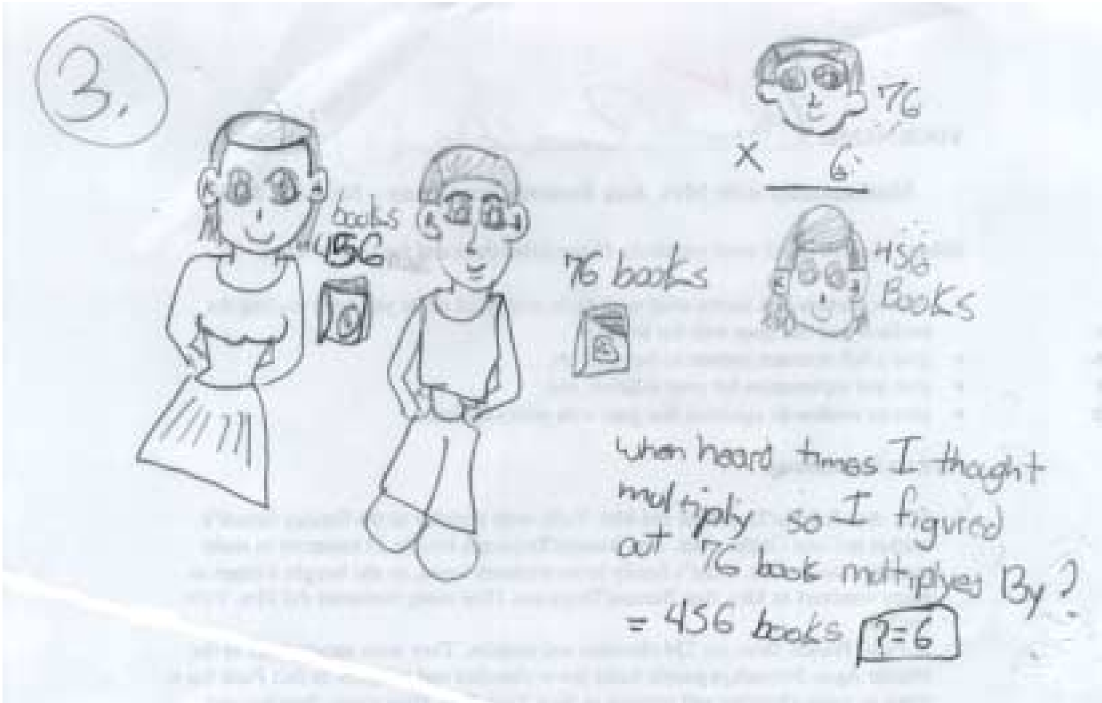
Mindy, problem 2: “I saw 234 churches and temples in Paris and then I saw New York with some but I couldn’t count it. Then I saw 234 churches and temples then I saw a church with a × sign and then I saw a temple with a 6 and a = sign and another church. Then I multiplied it and realized that it was not right because New York had more. So I did it again and again until I realized that I have to divide. I did and I got 39.”



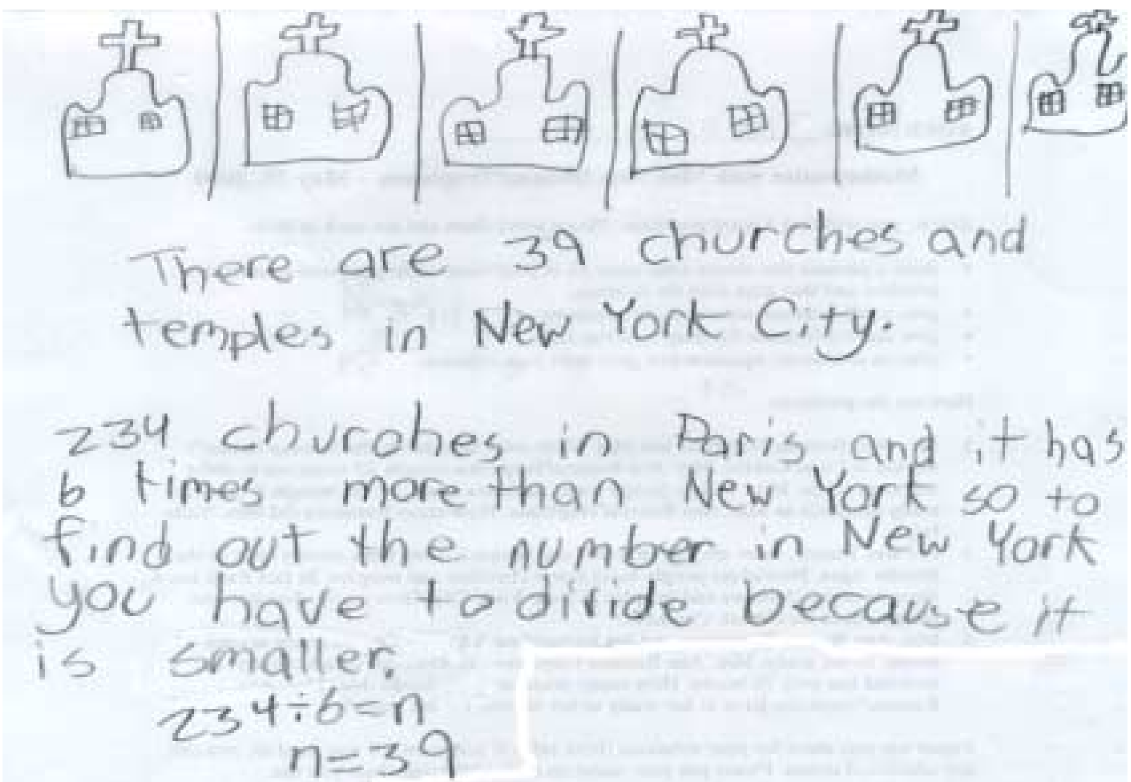
Ray, problem 1: “When I heard that Mrs. Ana bought 23 tomatoes and Mrs. Valle bought 6 times as more I think Times (Times Table multiply).”

Ray, problem 2: “When I heard times I thought multiply, but I’m making a smaller number, so you divide. $234 \div 6 = 39$.”

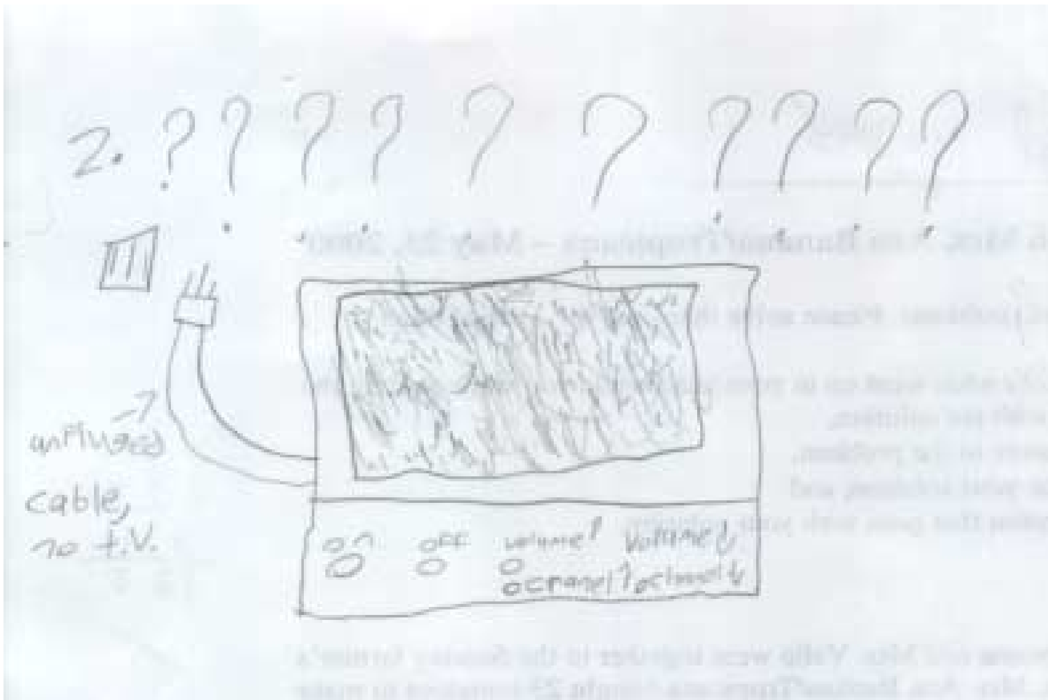
Ray, problem 3: “When I heard times I thought multiply, so I figured out 76 books multiplyes By ? = 456 books. ? = 6.”



Jay, problem 2: “234 churches in Paris and it has 6 times more than New York so to find out the number in New York you have to divide because it is smaller. 234÷ 6 = n. n = 39.”

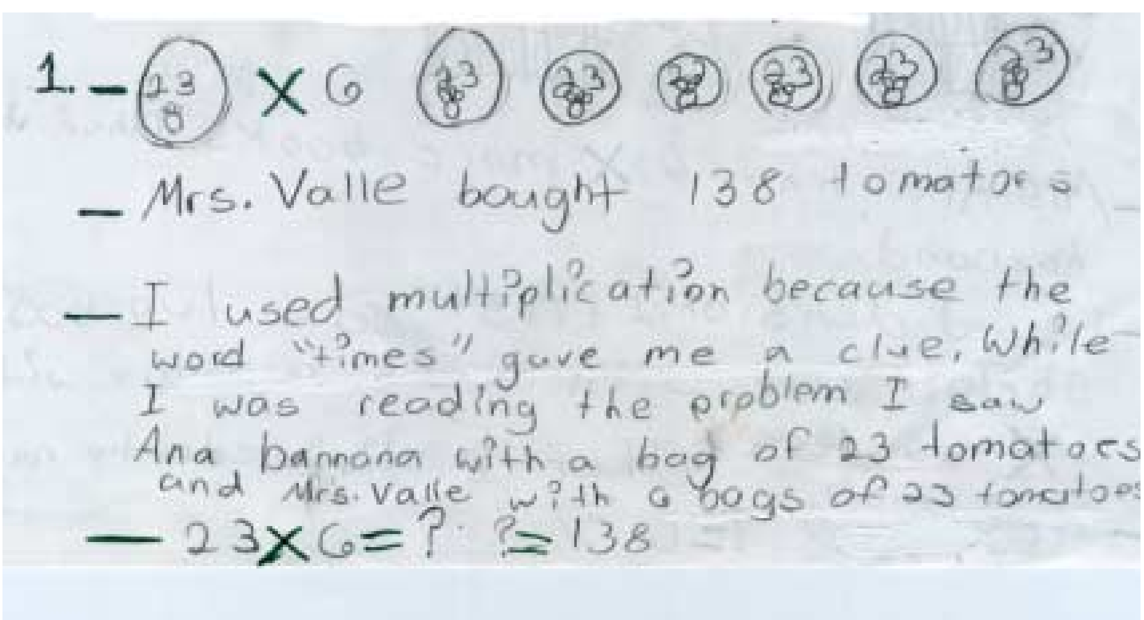


By utilizing SEL to share their sensory experiences, children who could not solve a problem were still able to shift to a resourceful state marked by humor, where it was quite easy for them to construct new learnings. An excellent example is Dave, who could not solve problem 2 and likened his mind to an unplugged TV set.

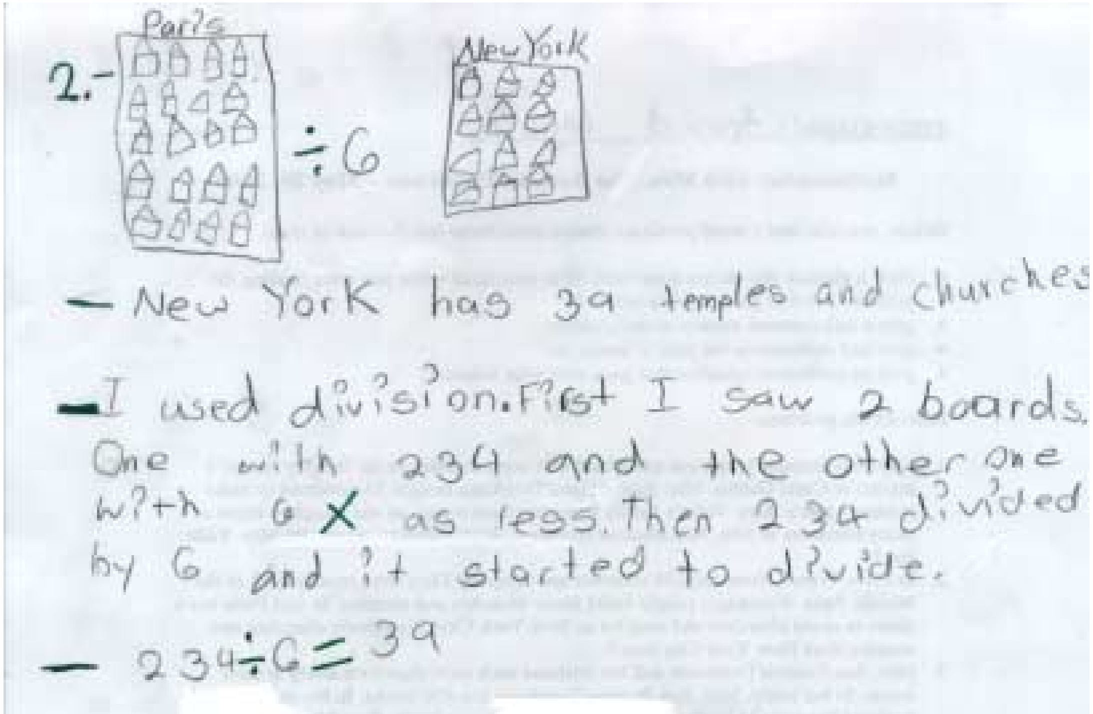


It is also remarkable how quickly the children started to use the term “times as less” in contrast to the terms “times as many” and “times as more,” to let them know that they had to divide, rather than multiply. Here are some examples:

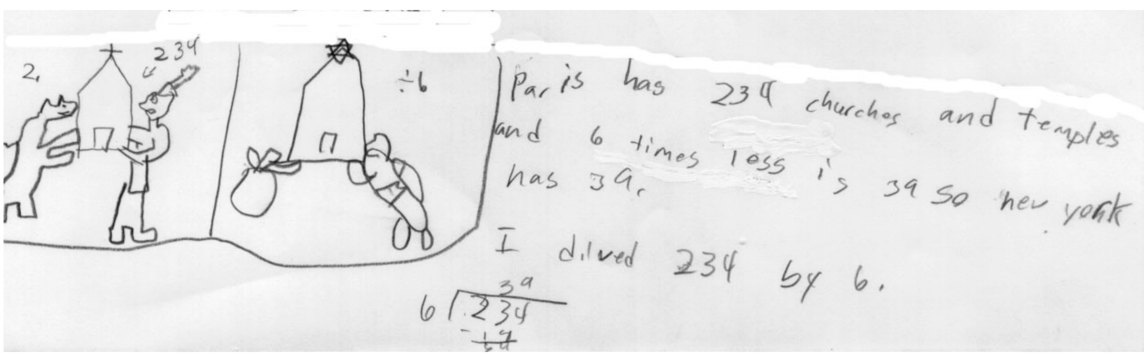
Erwin, problem 1: “I used multiplication because the word “times” gave me a clue. While I was reading the problem I saw Ana bannana with a bag of 23 tomatoes and Mrs. Valle with 6 bags of 23 tomatoes.” Note the clarity of the representation into 6 sets of equal size.



Erwin, problem 2: “I used division. First I saw 2 boards. One with 234 and the other one with 6 x as less. Then 234 divided by 6 and it started to divide. 234 ÷ 6 = 39.”



Ben, problem 2: “Paris has 234 churches and temples and 6 times less is 39 so New York has 39. I divided 234 by 6.”



Conclusion

In the past two decades a paradigm change has swept through the mathematics education research. The change has now reached the professional organizations and the principal granting agencies, which, in turn, are leading reform efforts to implement the new, constructivist principles in the mathematics classroom.

Backed up by a huge body of research of which we have given a few examples, the new 2000 Principles and Standards for School Mathematics of the National Council of Teachers of Mathematics (NCTM, 2000) now requires teaching mathematics for understanding and helping students build new knowledge from their experience and prior knowledge. In this paper we have dealt with the question of how to actually implement these requirements. In spite of increased efforts in mathematics education research to reconnect mathematics understanding to students' experiences, we have found that authors usually stay at an abstract level of information processing (such as the level of knowledge structures), thus missing the basic *sensory* level where actual meaning making is manifested in consciousness. They fail to literally "embody" reform efforts.

In this paper we have described an ongoing pilot project in elementary mathematics education that applies the methodology of a larger ongoing Linguistic Action Inquiry Project to help successfully root mathematical understanding in students' prior *sensory* experiences. The key to this methodology is SEL, a shared experiential language that allows a direct, two-way communication between the teachers and the students at a level where the students' individual meaning making is of highest priority. We have presented SEL, that comprises of categories of subjective experience, such as submodalities, sensory strategies, and physiological cues, as well as ways for the teachers to separate student's meanings from their own. The premise of our research in general, and our pilot project in particular, is that if the teacher/investigator *embodies* these categories of subjective experience in her neurology and mindfully reflects them in her communication with the students, then she is able to share the students' experiences at a deep sensory level and thus she is able to literally "make more sense" of her students. The fact that students respond so readily when the teacher/investigator starts looking at their process of mathematics understanding through the lens of these subjective experience categories, demonstrates that we have indeed created a shared language of experience. This language allows communication with the students to become a two-way process: in action inquiry cycles the teacher/investigator gets immediate feedback from the students on how they literally re-present, that is, make sense of mathematical communication, and so she is able to adjust her communication to *fit* their sense-making.

By rooting mathematics understanding in each student's individual sensory experiences, we are also shifting the responsibility for success in mathematics from the students back to those who guide and lead the process of co-constructing knowledge. This, in turn, should radically change prevailing beliefs "about who should be studying" mathematics and "who should be successful at it" (Doerr and Tinto, 1999, p. 424): Everybody has access to understanding, not just those who possess the "math gene"—it should not be socially acceptable any more to fail in mathematics.

In this paper we have also proposed ways of enhancing the teacher/investigator as a teaching/research tool to be able to embody the categories that make up SEL. It is also important to emphasize that in order for the teacher/investigator to become an exquisite teaching/research tool, she must first of all master mathematical *content knowledge*. This will give her the freedom to acquire *multiple* approaches to mathematics problem solving, be able to "try on" the students' perspectives, and make "good use of the learning opportunities created by children as they are engaged in interesting tasks" (Confrey, 1999, p. 96). Only too often teachers "miss or obliterate these opportunities repeatedly, seeking premature closure with the goals that they initially set. Consequently, children's 'wonderful ideas' (Duckworth, 1987) get overridden or ignored. This is the result of such problems as insufficient or, too often in the case of secondary and postsecondary teachers, inflexible knowledge of the subject matter, failure to engage students in motivating tasks, lack of belief in students' capacity, an overburdened curriculum, and/or an accountability in assessment systems that reward superficial behaviors" (Confrey, 1999, p. 96). What is needed is the implementation of new teacher enhancement programs where our methodology of enhancing the person of the teacher as the main teaching/research tool is interwoven with methods for increasing teachers' mathematics content knowledge and competence.

Confrey (1999) proposed "the creation of a centralized repository of examples of students' approaches edited carefully, replete with students' work," to be made "accessible to practicing teacher-educators and schools. The examples should illustrate not only students' typical approaches, but also the fertile possibilities that they raise, the challenges that children are deemed capable of meeting, and sociocultural views of what mathematics and science are, acknowledging diverse processes and perspectives" (p. 95). Our methodology of modeling students' sensory strategies seems to be an excellent tool for the creation of such a repository.

Utilizing SEL, in our project we model the students' subjective experience in the context of mathematics learning to help them amplify successful learning states by bringing them into consciousness, and, if necessary, we help them shift unresourceful learning states so that they become resourceful. Our attention is focused on how students experience mathematics learning and how to help them back up undesired experiences to a place where change happens easily. Our premise is that experiences are like a series of dominoes: the more dominoes are falling, the more difficult it is to break unuseful learning patterns. If we can find the first domino or what has knocked down the first domino, so to speak, then the person has much more choice than when his negative response—be it anger, frustration, or helplessness—is real high. It is much more likely that a student has choice while his response to a negative state of learning is still small, and it gives him a sense of control to be able to change it. Through the process of modeling students' experiences we slow down their processing so they are able to gain conscious control over their sensory strategies and thus gain conscious mathematical competence.

Acknowledgements

The first author hereby thankfully acknowledges valuable feedback from Raquel July and Cengiz Alacaci in the College of Education at Florida International University. They have read early drafts of the paper, and we have incorporated their comments in subsequent drafts. Cengiz, who teaches elementary mathematics education, also pointed Ana to resources for mathematical content teaching techniques. She also thanks her husband, Michael Lenart, for critically reading *all* drafts of the paper, and for helping insert samples of the children's homework.

References

Argyris, C., Putnam, R., & Smith, M. C. (1985*). Action science: Concepts, methods, and skills for research and intervention.* San Francisco: Jossey-Bass.

Argyris, C., & Schön, D. (1974). *Theory in practice: Increasing professional effectiveness.* San Francisco: Jossey-Bass.

Argyris, C., & Schön, D. (1978). *Organizational learning*. Reading, MA: Addison-Wesley.

Askey, R. (1999). Why does a negative ∗ a negative = a positive? *American Educator*, 23 (3), 1999, pp. 10–11.

Baars, B. J. (1988). *A cognitive theory of consciousness.* Cambridge: Cambridge University Press.

Bandler, Richard and MacDonald, Will. (1988). *An Insider's Guide to Sub-modalities*. Cupertino, Ca.: Meta Publications.

Bateson, G. (1972). *Steps to an Ecology of Mind*. Toronto, Canada: Balantine Books.

Bateson, M. C. (1972). Learning: An entry into pattern. In *Our own metaphor* (pp. 104–120). New York: Knopf.

Calvin W. (1990). *Cerebral symphony: Seashore reflections on the structure of consciousness*. New York: Bantam Books.

Chalmers, D. J. (1995). Facing up to the problem of consciousness. *Journal of Cosnciousness Studies*, **2** (3), 200–219.

Chalmers, D. J. (1996). *The conscious mind: In search of a fundamental theory*. New York: Oxford University Press.

Churchland, P.S., & Sejnowski, T. (1992). *The computational brain.* Cambridge, MA: MIT Press.

Cobb, Paul and Gravemeijer, Koeno and Yackel, Erna and McClain, Kay, and Whitenack, Joy (1997). Mathematizing and Symbolizing: The Emergence of Chains of Signification in One First-Grade Classroom. In Kirshner, David and Whitson, James A. (Eds.), *Situated Cognition: Social, Semiotic, and Psychological Perspectives* (pp. 151–234). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Confrey, J. (1999). Improving Research and Systemic Reform Toward Equity and Quality. In Kelly, A. E. and Lesh, R. A. (Eds.). *Handbook of Research Design in Mathematics and Science Education* (pp. 87–106). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Crabtree, B. F., and Miller, W. L. (1992). Doing qualitative research: Multiple strategies. Newbury Park, CA: Sage.

Crick, F. (1994). *The astonishing hypothesis.*New York: Scribners.

Csikszentmihalyi, M. (1978). Attention and the holistic approach to behavior. In K. S. Pope and J. L. Singer (Eds.), *The stream of consciousness* (pp. 335-358). New York: Plenum.

Csikszentmihalyi, M. (1990). *Flow: The psychology of optimal experience.* New York: Harper and Row.

Csikszentmihalyi, M. (1997). *Finding flow: The psychology of engagement with everyday life.* New York: Basic Books.

Damasio, Antonio R. (1994). *Descartes' Error*. New York: Grosset/Putnam.

Davis, R. B. and Maher, C. A. (1990). What do we do when we "do mathematics"? In R. B. Davis, C. A. Maher, C. A., and N. Noddings (Eds.), *Constructivist views on the teaching* *and learning of mathematics* (pp. 65–78). Monograph 4 of the *Journal for Research in Mathematics Education.* Reston, VA: National Council of Teachers of Mathematics.

Davis, Robert B. and Maher, Carolyn A. (1997). How students Think: The Role of Representations. In English, Lyn (Ed.), *Mathematical Reasoning. Analogies, Metaphors, and Images* (pp. 93–115). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Dennett, D. C. (1991). *Consciousness explained.* Boston: Little Brown.

Dilts, R. B., Dilts, R. W. and Epstein T. (1991). *Tools for Dreamers: Strategies for Creativity and the Structure of Innovation*. Cupertino, Ca.: Meta Publications.

Doerr, H. M. and Tinto, P. P. (1999). Paradigms for Teacher-Centered Classroom-Based Research. In Kelly, A. E. and Lesh, R. A. (Eds.). *Handbook of Research Design in Mathematics and Science Education* (pp. 403–427). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Dufour-Janvier, B., Bednarz, N., and Belanger, M. (1987). Pedagogical considerations concerning the problem of representation. In Janvier, Claude (Ed.) *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 109–122). Hillsdale, New Jersey: Lawrence Erlbaum Associates.

Edelman, G. (1989). *The remembered present: A biological theory of consciousness*. New York: Basic Books.

English, Lyn (Ed.) (1997). *Mathematical Reasoning. Analogies, Metaphors, and Images.* Mahwah, New Jersey: Lawrence Erlbaum Associates.

English, Lyn. (1997). Analogies, Metaphors, and Images: Vehicles for Mathematical Reasoning. In English Lyn (Ed.). (1997). *Mathematical Reasoning. Analogies, Metaphors, and Images* (pp. 4–18). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Erickson, M.H. (1958). Naturalistic techniques of hypnosis: Utilization techniques. *American Journal of Clinical Hypnosis,* **8,** 57–65.

Flanagan, O. (1992). *Consciousness reconsidered.* Cambridge, MA: MIT Press.

Gergen, K. J. (1991). *The Saturated Self*. New York: Basic Books.

Globus, G. (1995). *The post-modern brain.* New York: Benjamin.

Hale-Haniff, M. (1986). *Accessing the right brain.* Unpublished manuscript.

Hale-Haniff, M. (1989). Changeworker's guide to inter and intra-personal communication. Unpublished manuscript.

Hale-Haniff, M. and Pasztor, A. (1999), Co-constructing Subjective Experience: A Constructivist Approach. *Dialogues in Psychology*,16.0. http://hubcap.clemson.edu/psych/Dialogues/dialogues.html.

Hiebert, J. and Carpenter, T. P. (1992). Learning And Teaching With Understanding. In Grouws, D. A. (Ed.), *Handbook of Research on Mathematics Teaching and Learning: A Project of the National Council of Teachers of Mathematics* (pp. 65–97). New York: MacMillan .

Hoyt, M. F. (Ed.) (1994). *Constructive Therapies*. New York: The Guilford Press.

Jackendoff, R. (1987). *Consciousness and the computational mind.* Cambridge, MA: MIT Press.

Jackiw, N. (1995). The geometer's sketchpad (Macintosh Version 3) [Computer software]. Berkeley, CA: Key Curriculum Press.

Jansen, G. and Peshkin, A. (1992). Subjectivity in Qualitative Research. In LeCompte, M. D., Millroy, W. L. and Preissle, J. (Eds.). *The Handbook of Qualitative Research in Education* (pp. 681–725). San Diego, Ca: Academic Press. Chapter 15.

Johnson, M. (1987). *The body in the mind: The bodily basis of meaning, imagination, and reason.* Chicago: University of Chicago Press.

Kelly, A. E. and Lesh, R. A. (Eds.) (1999). *Handbook of Research Design in Mathematics and Science Education.* Mahwah, New Jersey: Lawrence Erlbaum Associates.

Kelly, A. E. and Lesh, R. A. (1999). Trends and Shifts in Research Methods. In Kelly, A. E. and Lesh, R. A. (Eds.). *Handbook of Research Design in Mathematics and Science Education* (pp. 35–44). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Lakoff, G. (1987). *Women, fire, and dangerous things: What categories reveal about the mind*. Chicago and London: University of Chicago Press.

Lakoff, G. and Johnson, M. (1999). *Philosophy in the Flesh*. Basic Books.

Lakoff, G. and Nuñez, R. E. (1997). Cognitive Foundations for a Mind-Based Mathematics. In English, Lyn (Ed.), *Mathematical Reasoning. Analogies, Metaphors, and Images* (pp. 21–92). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Lesh, R. A. and Lovitts, B. (1999). Research Agendas: Identifying Priority Problems and Developing Useful Theoretical Perspectives. In Kelly, A. E. and Lesh, R. A. (Eds.). *Handbook of Research Design in Mathematics and Science Education* (pp. 45–72). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Lesh, R. A., Lovitts, B., and Kelly, A. E. (1999). Purposes and Assumptions of This Book. In Kelly, A. E. and Lesh, R. A. (Eds.). *Handbook of Research Design in Mathematics and Science Education* (pp. 17–33). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Lincoln, Y. S. and Guba, E. G. (1985), *Naturalistic inquiry.* Newbury Park, CA: Sage Publications.

Machtinger, D. D. (1965). *Experimental course report: Kindergarten.* (Report No. 2, July, 1965) Webster Groves, MO: The Madison Project.

Maher, C. A., Davis, R. B., and Alston, A. (1992). Teachers paying attention to students' thinking. *Arithmetic Teacher,* 39(9), 34–57.

McGinn, C. (1991). *The problem of consciousness.* Oxford: Blackwell.

McNeill, R. (1988). A Reflection on When I Loved Math and How I Stopped. *Journal of Mathematical Behavior*, vol. 7, pp. 45–50.

Minsky, M. (1986). *The Society of Mind*. New York: Simon & Schuster.

Nagel, T. (1986). *The view from nowhere.* New York: Oxford University Press.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. http://standards.nctm.org/protoFINAL/

National Science Foundation (1999). Guide to Programs, 99–4.

Neimeyer, R. A. and Mahoney, M. J. (Eds.), (1995). *Constructivism in Psychotherapy*. Washington, D. C.: American Psychological Association.

NSF (1999). *Guiding Principles for Mathematics and Science Education Research Methods: Report of a Workshop*. Draft, 12/21/99. Division of Research, Evaluation and Communication, National Science Foundation.

Pasztor, A. (1998). Subjective Experience Divided and Conquered. *Communication and Cognition*, 31 (1), pp. 73–102.

Pert, C. B. (1997). *Molecules of Emotion*. New York: Scribner.

Presmeg, N. C. (1997). Generalization Using Imagery in Mathematics. In English, Lyn (Ed.), *Mathematical Reasoning. Analogies, Metaphors, and Images* (pp. 299–312). Mahwah, New Jersey: Lawrence Erlbaum Associates.

Reason, P. (1998). Three Approaches to Participative Inquiry. In Denzin, N. K. and Lincoln, Y. S. (Eds). *Strategies of Qualitative Inquiry* (pp. 261- 291). Thousand Oaks, Ca: Sage Publications.

Rubin, L. B. (1981). Sociological research: The subjective dimension. *Symbolic Interaction,* 4, pp. 97–112.

Satir, V. (1967). *Conjoint family therapy* (revised ed.). Palo Alto: Science and Behavior Books.

Schön, D. A. (1983). *The reflective practitioner: How professionals think in action*. New York: Basic Books.

Schwartz, P. and Ogilvy, J. (1979). *The emergent paradigm: Changing patterns of thought and belief* (SRI International). Cited in Lincoln, Y.S. and Guba, E. G. (1985), *Naturalistic inquiry.* Newbury Park, CA: Sage Publications.

Searle, John, R. (1992). *The Rediscovery of the Mind*. Cambridge, Ma.: MIT Press.

Simon, H. A. (1984). *The Sciences of the Artificial.* Cambridge, Ma: The MIT Press.

Steffe, L. P. (1977). *Constructivist models for children's learning in arithmetic*. Paper presented at the Research Workshop on Learning Models, Durham, NH.

Thagard, Paul. (1996). *Mind. Introduction to Cognitive Science*. Cambridge, Ma.: MIT Press.

Torbert, W. R. (1981). Why educational research has been so uneducational: The case for a new model of social science based on collaborative inquiry. In Reason, P. and Rowan, J. (Eds.), *Human inquiry: A sourcebook of new paradigm research* (pp. 141–152). Chichester, UK: John Wiley.

Torbert, W. R. (1991). *The power of balance: Transforming self, society, and scientific inguiry*. Newbury Park, CA: Sage.

Van de Walle, J. (1998). *Elementary and Middle School Mathematics. Teaching Developmentally.* Addison Wesley Longman.

Varela, F. (1996). "The specious present": A neurophenomenology of nowness. In J. Petitot , J. M. Roy, B. Pachoud, & F. Varela (Eds.), *Contemporary issues in phenomenology and cognitive science.* Stanford: Stanford University Press.

Varela, F. (1996a). Neurophenomenology: A methodological remedy for the hard problem. *Journal of Consciousness Studies*, **3**(4), 330–349.

Varela, F., Thompson, E., & Rosch, E. (1991). *The embodied mind: Cognitive science and human experience.* Cambridge, MA: The MIT Press.

Vico, G. (1948). *The new science* (T. G. Bergin & M. H. Fisch, Trans.). Ithaca, NY: Cornell University Press. (Original work published in 1725.)

Von Glasersfeld, Ernst (1987). Learning as a Constructive Activity. In Janvier, Claude (Ed.) *Problems of Representation in the Teaching and Learning of Mathematics* (pp. 3 – 18). Hillsdale, New Jersey: Lawrence Erlbaum Assoc.

Watzlawick, P. (1984). Components of Ideological "Realities." In Watzlawick, P., Ed., *The Invented Reality* (pp. 206–248). New York: W. W. Norton & Company.

Wheatley, G. H. (1997). Reasoning With Images in Mathematical Activity. In English, Lyn (Ed.), *Mathematical Reasoning. Analogies, Metaphors, and Images* (pp. 281–297). Mahwah, New Jersey: Lawrence Erlbaum Associates.