COT 3420
Section U02
Spring 2012

EXAM # 1 ANSWERS

QUESTIONS

Question 1. (30 points)

1. c 2. c 3. b 4. c 5. c 6. c 7. c 8. d 9. a 10. d 11. c 12. b 13. b 14. c 15. a

Grading Criteria: 2 points for each correct answer.

Question 2. (20 points)
Proof: First we notice that $< a, b > \in R^n$ if there is a path of length n from a to n,

$a \longrightarrow c_1 \longrightarrow c_2 \longrightarrow \ldots \longrightarrow c_n = b$

The problem asks us to prove the 3 statements below.

1. R^+ is transitive
2. R^+ contains R
3. if S is transitive and $R \subset S$, then $R^+ \subset S$

1. We start with 1.

Let $< a, b >, < b, c > \in R^+$. Then there are m, n such that $< a, b > \in R^m$ and $< b, c > \in R^n$. Then $< a, c > \in R^{m+n}$. Since $R^{m+n} \subset R^+$, $< a, c > \in R^+$.

2. This is satisfied since $R \subset R^+$.

3. Let S be a transitive relation that contains R. We will show that for all $n \geq 1$, $R^n \subset S$.

We can do by induction on n.

Basis: $n = 1$ $R \subset S$ by assumption.

Inductive Step: Assume that $R^n \subset S$. Since $R \subset S$, $R^n \circ R \subset S \circ R \subset S \circ S$. Now, S is transitive, so $S \circ S \subset S$. We get that $R^n \circ R \subset S$. Since $R^n \circ R = R^{n+1}$, $R^{n+1} \subset S$.

Now, that we showed that each $R^n \subset S$. Then we get that their union, $R^+ = R \cup \ldots R^n \ldots \subset S$.

grading Criteria:

1. Just trying: 2 points
2. Showing That R^+ is transitive : 6 points
3. Showing that R^+ is the least upper bound: 11 points
4. Misc. 1point

Question 3. (30 points)

1. b 2. c 3. a 4. c 5. a 6. c 7. c 8. b 9. c 10. b 11. c 12. d 13. d 14. d 15. a

Grading Criteria: 2 points for each correct answer.

Question 4. (20 points)

Case 1: $F = P_i$ for some $i \in N$. Since P_i is an atom, $|F| = 1$ and $n[con, F] = n[\neg, F] = 0$. The equality becomes

$1 = 4 * 0 + 0 + 1$

which is true.

Case 2: $F = \neg G$.

By IH,

(IH) $|G| = 4 * n[con, G] + n[\neg, G] + 1$

Now we relate the length and the F-counts to the length and the G-counts.

(1) $|F| = |G| + 1$

(2) $n[con, F] = n[con, G]$

(3) $n[\neg, F] = n[\neg, G] + 1$

Now,

$|F| = |G| + 1$ by (1)

$= (4 * n[con, G] + n[\neg, G] + 1) + 1$ by (IH)

$= 4 * n[con, G] + (n[\neg, G] + 1) + 1$ by grouping

$= 4 * n[con, F] + (n[\neg, G] + 1) + 1$ by (2)

$= 4 * n[con, F] + n[\neg, F] + 1$ by (3)

So we got $|F| = 4 * n[con, F] + n[\neg, F] + 1$.

Case 3: $F = (GCH)$ where C is a binary connective.

By IH on G and H we have:

(IH1) $|G| = 4 * n[con, G] + n[\neg, G] + 1$

(IH2) $|H| = 4 * n[con, H] + n[\neg, h] + 1$

Now we relate the length and the F-counts to the lengths and the counts of G and H.

(4) $|F| = |G| + |H| + 3$

(5) $n[con, F] = n[con, G] + n[con, H] + 1$

(6) $n[\neg, F] = n[\neg, G] + n[\neg, H]$

Now we relate $|F|$ to the F counts.

$|F| = |G| + |H| + 3$ by (4)

$= (4 * n[con, G] + n[\neg, G] + 1) + |H| + 3$ by (IH1)

$= (4 * n[con, G] + n[\neg, G] + 1) + (4 * n[con, H] + n[\neg, H] + 1) + 3$ by (IH2)

$= (4*n[con, G]+4*n[con, H]+4)+(n[\neg, G]+n[\neg, H])+1$ by grouping

$= 4*(n[con, G]+n[con, H]+1)+(n[\neg, G]+n[\neg, H])+1$ by distributivity

$= 4 * n[con, F] + (n[\neg, G] + n[\neg, H]) + 1$ by (5)

$= 4 * n[con, F] + n[\neg, F] + 1$ by (6)

So we got $|F| = 4 * n[con, F] + n[\neg, F] + 1$.

Grading Criteria:

1. Listing the cases: 2 points
2. Case 1: 2 points
3. Case 2: 6 points
    - 3.1: the IH: 1 point
    - 3.2: formulas (1)-(3): 2 points
    - 3.3: the derivation: 3 points
        - 3.3.1: the explanation of the derivation: 1 point
4. Cases 3-6: 10 points
    - 4.1: the IH: 2 points
    - 4.2: formulas (4)-(6): 3 points
    - 4.3: the derivation: 5 points
        - 4.3.1: the explanation of the derivation: 1 point