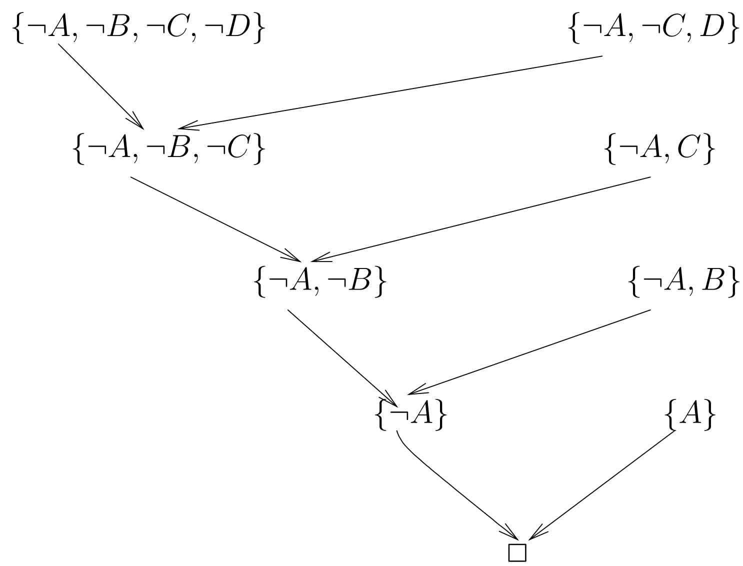


Figure 1: The resolution tree for Question 2

**COT 3420**
**Section U02**
**Spring 2012**

## EXAM # 2 ANSWERS

**Question 1.** (20 points)
1. b 2. b 3. a 4. a 5. c 6. d 7. c 8. c 9. c 10. c

**Grading Criteria:** 2 points for each correct answer.

**Question 2.** (15 points)
Construct a derivation tree of $\Box$ from
$S = \{\{\neg A, \neg B, \neg C, \neg D\}, \{\neg A, C\}, \{\neg A, \neg C, D\}, \{A\}, \{\neg A, B\}\}$.

The answer is in Figure 1.

**Grading Criteria:** 1. 15/4 points for each correct resolution step (up to 4).

2. -4 points for each incorrect resolution step.

3. - 1.5 points for each extra or useless resolution step.

**Question 3.** (20 points)

Proof or Disproof. If $F \longrightarrow G$ is satisfiable and $H$ is a formula, then $(F \wedge H) \longrightarrow (G \wedge H)$ is satisfiable.

First write if you go for proof or disproof. Then provide you proof or the counter-example.

**Proof**. Assume that $\mathcal{A}$ is a model of $F \longrightarrow G$. We have 2 cases, according to whether $\mathcal{A}[H] = 0$ or $\mathcal{A}[H] = 1$.

Case 1: $\mathcal{A}[H] = 0$

$\mathcal{A}[(F \wedge H) \longrightarrow (G \wedge H)] = \mathcal{A}[F \wedge H] \longrightarrow \mathcal{A}[G \wedge H]$ interpretation of $\longrightarrow$

$= [\mathcal{A}[F] \wedge \mathcal{A}[H]] \longrightarrow [\mathcal{A}[G] \wedge \mathcal{A}[H]]$ interpretation of $\wedge$

$= [\mathcal{A}[F] \wedge 0] \longrightarrow [\mathcal{A}[G] \wedge 0]$ $\mathcal{A}[H] = 0$

$= 0 \longrightarrow 0$ table of $\wedge$

$= 1$ table of $\longrightarrow$

So, $\mathcal{A}[(F \wedge H) \longrightarrow (G \wedge H)] = 1$.

Case 2: $\mathcal{A}[H] = 1$

$\mathcal{A}[(F \wedge H) \longrightarrow (G \wedge H)] = \mathcal{A}[F \wedge H] \longrightarrow \mathcal{A}[G \wedge H]$ interpretation of $\longrightarrow$

$= [\mathcal{A}[F] \wedge \mathcal{A}[H]] \longrightarrow [\mathcal{A}[G] \wedge \mathcal{A}[H]]$ interpretation of $\wedge$

$= [\mathcal{A}[F] \wedge 1] \longrightarrow [\mathcal{A}[G] \wedge 1]$ $\mathcal{A}[H] = 1$

$= \mathcal{A}[F] \longrightarrow \mathcal{A}[G]$ table of $\wedge$

$= \mathcal{A}[F \longrightarrow G]$ interpretation of $\longrightarrow$

$= 1$ since $\mathcal{A}$ is a model of $F \longrightarrow G$

So, $\mathcal{A}[(F \wedge H) \longrightarrow (G \wedge H)] = 1$.

**Grading Criteria:**

1. If you wrote disproof: 3 points.
2. If you wrote proof 6 points plus the points on the proof.
3. The split into 2 cases: 3 points.
4. Each case is worth 5.5 points, 2 points of which are the reasons.

**Question 4.** (20 points)

a. (10 points) Prove that the set of connectives $U = \{F \wedge \neg G\}$ is not adequate.

b. (10 points) Use the fact that $S = \{F \wedge G, \neg F\}$ is adequate to show that $V = \{F \wedge \neg G, \mathbf{T}\}$ is adequate, where $\mathbf{T}$ is a tautology.

a. Let $\mathcal{E}$ be the truth assignment that assigns 0 to every atom. We'll show that $\mathcal{E}$ is a countermodel for every $U$-formula. For simplicity we write $\phi(F, G)$ instead of $F \wedge \neg G$. The proof is by structural induction on $U$.
Case 1: $F$ is an atom. Then $\mathcal{E}[F] = 0$, so $\mathcal{E}$ is a counter-model of $F$.

Case 2: $F = \phi(G, H)$. Then,
$\mathcal{E}[F] = \mathcal{E}[\phi(G, H)]$ $\quad F = \phi(G, H)$
$= \mathcal{E}[G \wedge \neg H]$ $\quad \phi(F, G) = F \wedge \neg G$
$= \mathcal{E}[G] \wedge \mathcal{E}[\neg H]$ $\quad$ interpretation of $\wedge$
$= 0 \wedge \mathcal{E}[\neg H]$ $\quad$ IH on $G$
$= 0$ $\quad$ the table of $\wedge$
So, $\mathcal{E}[F] = 0$, i.e. $\mathcal{E}$ is a counter-example of $F$.

So, we showed that every $U$ formula has $\mathcal{E}$ as a counter model. Then $\mathbf{T}$, the tautology, has no equivalent $U$-formula.
**Grading Criteria:** 1. Finding an invariant for all $U$-formulas (in our case $\mathcal{E}$) : 4 points
2. Proving that it is an invariant : 6 points
3. Just trying: 1.5 points

b. We will implement every connective of $S = \{F \wedge G, \neg F\}$ with the connective $\phi[F, G] = F \wedge \neg G$ and the tautology $\mathbf{T}$.
Part 1. $\neg F \equiv \mathbf{T} \wedge \neg F$ $\quad$ tautology law
$= \phi[\mathbf{T}, F]$ $\quad$ definition of $\phi$
The last meta-formula is $V$.
Part 2. $F \wedge G \equiv F \wedge \neg\neg G$ $\quad$ double nagation elimination
$= \phi[F, \neg G]$ $\quad$ definition of $\phi$
$\equiv \phi[F, \phi[\mathbf{T}, G]]$ $\quad$ part 1
The last meta-formula is $V$.
**Grading Criteria:** 1. Part 1: 5 points
2. Part 2: 5 points
3. If you used strucrural induction on FORM $= \{\neg F, F \vee G, F \wedge G, F \longrightarrow G, F \longleftrightarrow G\}$, you loose 3 points.

4. Just Trying: 1.5 points

**Question 5** (25 points)

Apply the algorithm given in the book to find a CNF for the formula
$F = \neg[(A \vee B \vee \neg C) \longleftrightarrow \neg(\neg B \wedge C \wedge \neg D)]$.

**Solution**

$F = \neg[(A \vee B \vee \neg C) \longleftrightarrow \neg(\neg B \wedge C \wedge \neg D)]$ line 1

$\equiv \neg\{[(A \vee B \vee \neg C) \longrightarrow \neg(\neg B \wedge C \wedge \neg D)] \wedge [\neg(\neg B \wedge C \wedge \neg D) \longrightarrow (A \vee B \vee \neg C)]\}$ $\longleftrightarrow$-elim, line 2

$\equiv \neg\{[\neg(A \vee B \vee \neg C) \vee \neg(\neg B \wedge C \wedge \neg D)] \wedge [\neg\neg(\neg B \wedge C \wedge \neg D) \vee A \vee B \vee \neg C]\}$ $\longrightarrow$-elim twice, line 3

$\equiv \neg[\neg(A \vee B \vee \neg C) \vee \neg(\neg B \wedge C \wedge \neg D)] \vee \neg[\neg\neg(\neg B \wedge C \wedge \neg D) \vee A \vee B \vee \neg C]$ DeMorgan's, line 4

$\equiv [\neg\neg(A \vee B \vee \neg C) \wedge \neg\neg(\neg B \wedge C \wedge \neg D)] \vee [\neg\neg\neg(\neg B \wedge C \wedge \neg D) \wedge \neg A \wedge \neg B \wedge \neg\neg C]$ DeMorgan's, generalized DeMorgan's, line 5

$\equiv [(A \vee B \vee \neg C) \wedge \neg B \wedge C \wedge \neg D)] \vee [\neg(\neg B \wedge C \wedge \neg D) \wedge \neg A \wedge \neg B \wedge C]$ $\neg\neg$-elim 4 times, line 6

$\equiv [(A \vee B \vee \neg C) \wedge \neg B \wedge C \wedge \neg D)] \vee [\neg(\neg B \wedge C \wedge \neg D) \wedge \neg A \wedge \neg B \wedge C]$ $\neg\neg$-elim 4 times, line 7

$\equiv [(A \vee B \vee \neg C) \wedge \neg B \wedge C \wedge \neg D)] \vee [(\neg\neg B \vee \neg C \vee \neg\neg D) \wedge \neg A \wedge \neg B \wedge C]$ DeMorgan's, line 8

$\equiv [(A \vee B \vee \neg C) \wedge \neg B \wedge C \wedge \neg D)] \vee [(B \vee \neg C \vee D) \wedge \neg A \wedge \neg B \wedge C]$ $\neg\neg$-elim 2 times, line 9

$\equiv (A \vee B \vee \neg C \vee B \vee \neg C \vee D) \wedge (A \vee B \vee \neg C \vee \neg A) \wedge (A \vee B \vee \neg C \vee \neg B) \wedge (A \vee B \vee \neg C \vee C) \wedge (\neg B \vee B \vee \neg C \vee D) \wedge (\neg B \vee \neg A) \wedge (\neg B \vee \neg B) \wedge (\neg B \vee C) \wedge (C \vee B \vee \neg C \vee D) \wedge (C \vee \neg A) \wedge (C \vee \neg B) \wedge (C \vee C) \wedge (\neg D \vee B \vee \neg C \vee D) \wedge (\neg D \vee \neg A) \wedge (\neg D \vee \neg B) \wedge (\neg D \vee C)$ Generalized distributivity, line 10

$\equiv (A \vee B \vee \neg C \vee B \vee \neg C \vee D) \wedge (\neg B \vee \neg A) \wedge (\neg B \vee \neg B) \wedge (\neg B \vee C) \wedge (C \vee \neg A) \wedge (C \vee \neg B) \wedge (C \vee C) \wedge (\neg D \vee \neg A) \wedge (\neg D \vee \neg B) \wedge (\neg D \vee C)$ tautology elim, line 11

$\equiv (A \vee B \vee \neg C \vee D) \wedge (\neg A \vee \neg B) \wedge \neg B \wedge (\neg B \vee C) \wedge (\neg A \vee C) \wedge (\neg B \vee C) \wedge C \wedge (\neg A \vee \neg D) \wedge (\neg B \vee \neg D) \wedge (C \vee \neg D)$ idempotency, ordering the clauses, line 12

$\equiv (A \vee B \vee \neg C \vee D) \wedge \neg B \wedge C \wedge (\neg A \vee \neg D)$ absorbtion, line 13

**Note** If we use the reduction $(F \vee \neg G) \wedge G \implies F \wedge G$, then the above formula reduces to $(A \vee D) \wedge \neg B \wedge C \wedge (\neg A \vee \neg D)$ and we can simplify the computation quite a lot. For example, $[(A \vee B \vee \neg C) \wedge \neg B \wedge C \wedge \neg D)]$ reduces to $[A \wedge \neg B \wedge C \wedge \neg D)]$ and $[(B \vee \neg C \vee D) \wedge \neg A \wedge \neg B \wedge C]$ reduces to $[D \wedge \neg A \wedge \neg B \wedge C]$.

**Grading Criteria:** You get credit to the first line where you made a mistake.

1. If you got the first line correct you have 1 point.
2. If you got the first 2 lines correct you have 2 points.
3. If you got the first 3 lines correct you have 4 points.
4. If you got the first 4 lines correct you have 6 points.
5. If you got the first 5 lines correct you have 8 points.
6. If you got the first 6 lines correct you have 10 points.
7. If you got the first 7 lines correct you have 12 points.
8. If you got the first 8 lines correct you have 14 points.
9. If you got the first 9 lines correct you have 15 points.
10. If you got the first 10 lines correct you have 19 points.
11. If you got the first 11 lines correct you have 22 points.
12. If you got the first 12 lines correct you have 23 points.
13. If you got all 13 lines correct you have 25 points.
14. Not separating formulas by $\equiv$ : -2 points
15. Using parentheses around atoms or negations: -2 points
16. If your computation is correct but you started with the wrong formula, -3 points