COT 3420
Section U1
Summer 2005

THE ANSWERS TO EXAM# 1

QUESTIONS

Question 1. (36 points)
1. a 2. d 3. a 4. b 5. c 6. c 7. b 8. d 9. c 10. c 11. a 12. b

Grading Criteria
3 points for each correct answer.

Question 25. (25 points)
Prove that for all formulas F, $|F| = n[\neg, F] + 4 * n[atom, F] - 3$. Recall that $n[atom, F]$ is the number of atom occurrences in F and $|F|$ is the length of F.

Proof: By structural induction on F.

Case 1: F is an atom, say P_i. Since P_i is a symbol, $|P_i| = 1$. The symbol P_i is an atom and is not a negation. So, $n[atom, P_i] = 1$, and $n[\neg, P_i] = 0$. So, the equation $|F| = n[\neg, F] + 4 * n[atom, F] - 3$, reduces to
1 = 0 + 4 * 1 - 3
which is true.

Case 2: $F = \neg G$.
By induction hypothesis,
(IH) $|G| = n[\neg, G] + 4 * n[atom, G] - 3$
At the same time,
(1) $|F| = |G| + 1$
and
(2) $n[atom, F] = n[atom, G]$
(3) $n[\neg, F] = n[\neg G] + 1$
Now,
$|F| = |G| + 1$ by (1)
$= n[\neg, G] + 4 * n[atom, G] - 3 + 1$ by (IH)
$= (n[\neg, G] + 1) + 4 * n[atom, G] - 3$ by grouping
$= n[\neg, F] + 4 * n[atom, G] - 3$ by (3)

$= n[\neg, F] + 4 * n[atom, F] - 3$ by (2)

So, $|F| = n[\neg, F] + 4 * n[atom, F] - 3$

Case 3: $F = (GCH)$ where C is a binary connective, and G,H are formulas.

We apply the IH to G and H,

(IH1) $|G| = n[\neg, G] + 4 * n[atom, G] - 3$

(IH2) $|H| = n[\neg, H] + 4 * n[atom, H] - 3$

Now we relate the F counts to the G and H counts.

(4) $|F| = 1 + |G| + 1 + |H| + 1$

(5) $n[\neg, F] = n[\neg, G] + n[\neg, H]$

(6) $n[atom, F] = n[atom, G] + n[atom, H]$

Now,

$|F| = 1 + |G| + 1 + |H| + 1$ by (4)

$= |G| + |H| + 3$ by collecting the terms

$= n[\neg, G] + 4 * n[atom, G] - 3 + |H| + 3$ by IH1

$= n[\neg, G] + 4 * n[atom, G] + |H|$ by collecting the terms

$= n[\neg, G] + 4 * n[atom, G] + n[\neg, H] + 4 * n[atom, H] - 3$ by IH2

$= (n[\neg, G] + n[\neg, H]) + 4 * (n[atom, G] + n[atom, H]) - 3$ grouping and distributivity

$= n[\neg, F] + 4 * (n[atom, G] + n[atom, H]) - 3$ by (5)

$= n[\neg, F] + 4 * n[atom, F] - 3$ by (6).

Grading Criteria

1. Listing the cases: 2 points
2. Case 1: 3 points
3. Case 2: 8 points

3.1. Listing the IH: 2 points

3.2. Formulas 1, 2, 3 : 2 points

3.3. The derivation: 3 points

3.4. The reasons for the derivations: 1 point

4. Cases 3,4,5,6: 12 points

4.1. Listing the IH: 3 points

3.2. Formulas 3,4, 5 : 3 points

3.3. The derivation: 5 points

3.4. The reasons for the derivations: 1 point

Question 3. (15 points)

Draw the formula tree of $F = ((((P_0 \vee \neg P_1) \wedge \neg P_3) \longrightarrow ((P_3 \longrightarrow P_1) \longleftrightarrow \neg(P_5 \vee P_2))) \longleftrightarrow \neg((P_3 \wedge P_4) \vee (P_7 \longrightarrow \neg(P1 \wedge P_2))))$.

The tree is shown in Figure 1.

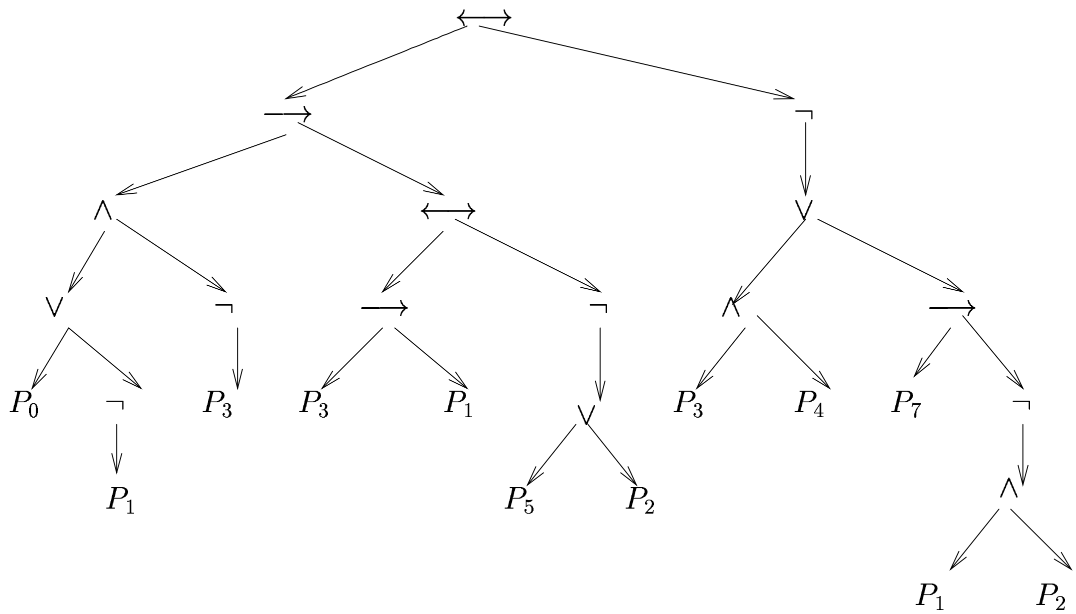

Figure 1: The tree for Question 3

Grading Criteria: -2 points for each wrong label or arrow.

Question 4. (15 points)

Write the formula that corresponds to the tree from Figure 2.

Write your answer below.

$\neg(((P_1 \longleftrightarrow (P_2 \wedge (P_3 \wedge P_3))) \vee \neg P_4) \longleftrightarrow ((\neg P_5 \vee (\neg P_2 \wedge P_1)) \wedge (\neg P_6 \longrightarrow (P_4 \longleftrightarrow (P_4 \vee P_1)))))$

Grading Criteria: -2 points for each out of place symbol.

Question 5. (25 points)

Prove or disprove: If F and G have no atoms in common, $F \longrightarrow H$ and $G \longrightarrow H$ are satisfiable, then $(F \vee G) \longrightarrow H$ is satisfiable.

Proof: We assume that

(1) $\models_{\mathcal{A}} (F \longrightarrow H)$

and

(2) $\models_{\mathcal{B}} (G \longrightarrow H)$

Case 1: H is satisfiable.

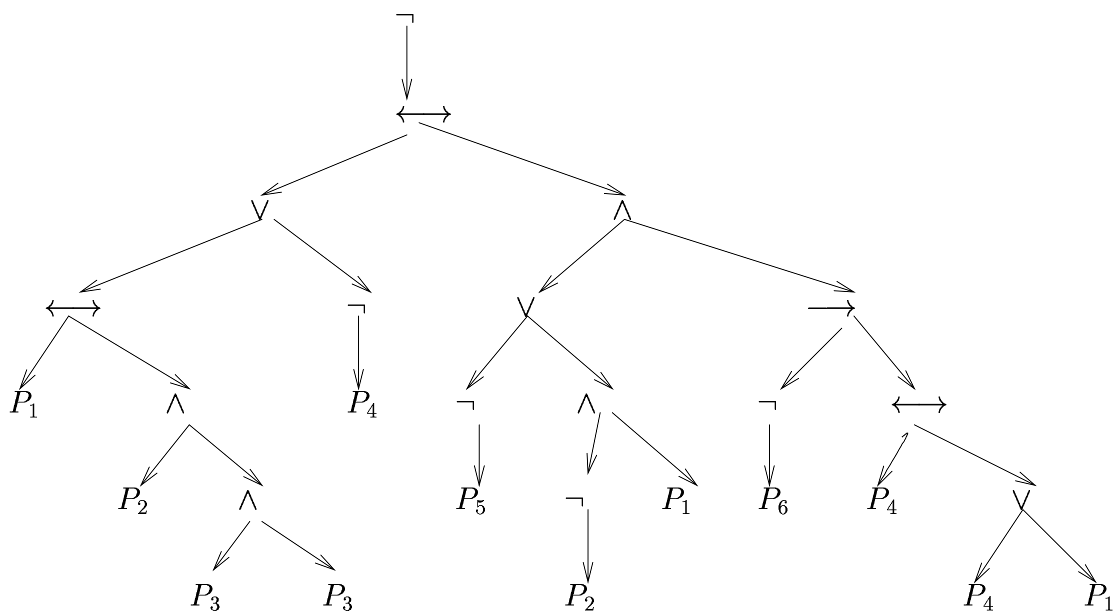

Figure 2: The tree for Question 4

Then H has a model, say $\mathcal{D}$. We will show that $\mathcal{D}$ is a model of $(F \vee G) \longrightarrow H$.

$\mathcal{D}[(F \vee G) \longrightarrow H] = \mathcal{D}[F \vee G] \boxed{\longrightarrow} \mathcal{D}[H]$

$= \mathcal{D}[F \vee G] \boxed{\longrightarrow} 1$ since $\mathcal{D}[H] = 1$

$= 1$ from the table of $\boxed{\longrightarrow}$

Case 2: H is unsatisfiable.

Let $\mathcal{C}$ be the truth assignment defined as

$\mathcal{C}[P_i] = \mathcal{A}[P_i]$ if P_i is in F

and

$\mathcal{C}[P_i] = \mathcal{B}[P_i]$ if P_i is not in F

Since F and G have no atoms in common, $\mathcal{C}$ agrees with $\mathcal{A}$ on F and $\mathcal{B}$ on G. We get (3) and (4) by The Agreement Theorem.

(3) $\mathcal{C}[F] = \mathcal{A}[F]$

(4) $\mathcal{C}[G] = \mathcal{B}[G]$

Since $\mathcal{A}[H] = 0$, (1) implies

(5) $\mathcal{A}[F] = 0$.

Since $\mathcal{B}[H] = 0$, (2) implies

(6) $\mathcal{B}[G] = 0$

We compute $\mathcal{C}[F \vee G]$.

$\mathcal{C}[F \vee G] = \mathcal{C}[F]\boxed{\vee}\mathcal{C}[G]$ interpretation of $\vee$

$= \mathcal{A}[F]\boxed{\vee}\mathcal{B}[G]$ from (3) and (4)

$= 0\boxed{\vee}0$ from (5) and (6)

$= 0$ from the table of $\boxed{\vee}$

So,

(7) $\mathcal{C}[F \vee G] = 0$

Then

$\mathcal{C}[(F \vee G) \longrightarrow H] = \mathcal{C}[F \vee G]\boxed{\longrightarrow}\mathcal{C}[H]$

$= 0\boxed{\longrightarrow}\mathcal{C}[H]$ by (7)

$= 1$ from the table of $\boxed{\longrightarrow}$

Grading Criteria:

1. If you didn't write proof or disproof or you wrote both: 0 points
2. If you wrote disproof: 3 points
3. If you wrote proof: 8 points + the points for the proof
4. Case 1 of the proof: 4 points
5. Case 2 of the proof : 13 points
6. The construction of $\mathcal{C}$: 8 points
7. The proof that $\mathcal{C}$ is a model for Case 2: 5 points.