

COT 3420
 Section U1
 Summer 2005

FINAL EXAM ANSWERS

Question 1. (10 points)

Rectify the formula

$$F = \forall x(P(f(x), y) \vee \forall y(\exists xP(x, y) \vee \neg \forall zP(x, g(y)))) \wedge \forall y(\exists x \neg P(y, x) \vee \neg P(y, x)).$$

$$F \equiv \forall x_1(P(f(x_1), y) \vee \forall y_1(\exists x_2P(x_2, y_1) \vee \neg P(x_1, g(y_1)))) \wedge \forall y_2(\exists x_3 \neg P(y_2, x_3) \vee \neg P(y_2, x)).$$

2 points for each relabeling x_1, x_2, x_3, y_1, y_2 .

Question 2 (10 points)

Skolemize the formula

$$F = \exists x \forall y(P(x, y) \vee \exists zP(z, f(y))) \wedge \forall u[\forall v(\exists w(\neg P(u, w) \vee \neg P(w, v)) \wedge \exists w_1 \neg P(w_1, u) \vee \exists w_2 \neg P(w_2, u))]$$

Solution

$$F \equiv_s \forall y(P(a, y) \vee P(g(y), f(y))) \wedge \forall u[\forall v((\neg P(u, h(u, v)) \vee \neg P(h(u, v), v)) \wedge \neg P(i(u, v), u) \vee \neg P(j(u), u))]$$

Grading Criteria

2 points for each of the 5 changes (x, z, w, w_1, w_2) .

Question 3. (20 points)

Prove by structural induction that every formula F has a prenex form. You may use the relabeling lemma and the semantic equivalences from the book, but do not use the Skolemization algorithm.

Proof

Case 1: F is an atomic formula.

Then F has no quantifiers, so F is a prenex form.

Case 2: $F = \neg G$.

By induction hypothesis G has a prenex form $Q_1x_1 \dots Q_nx_nI$, where I has no quantifiers.

Then, by repeatedly apply the equivalence $\neg QxH \equiv \overline{Q}x\neg H^1$, we get

$$F \equiv \overline{Q_1}x_1 \dots \overline{Q_n}x_n \neg I,$$

$${}^1\overline{\forall} = \exists, \overline{\exists} = \forall$$

and I has no quantifiers. The above formula is a prenex form.

Case 3: $F = G \vee H$.

By IH, G and H have prenex forms, $G \equiv Q_1x_1 \dots Q_nx_nI$ and $H \equiv q_1y_1 \dots q_my_mJ$, where the letters Q and q denote quantifiers, and I and J have no quantifiers. We use The Relabeling Lemma to change the y_i 's that occur in the prenex form of G .

We get $H \equiv q_1v_1 \dots q_mv_mJ[y_1/v_1, \dots, y_m/v_m]$.

Then,

$$\begin{aligned} F &\equiv Q_1x_1 \dots Q_nx_nI \vee q_1v_1 \dots q_mv_mJ[y_1/v_1, \dots, y_m/v_m] \\ &\equiv Q_1x_1 \dots Q_nx_nq_1v_1 \dots q_mv_m(I \vee J[y_1/v_1, \dots, y_m/v_m]) \end{aligned}$$

where we applied the rewrites

$$QwU \vee V \implies Qw(U \vee V)$$

and

$$U \vee qwV \implies qw(U \vee V)$$

These rewrites are equivalences when w does not occur in the other subformula.

Case 4: $F = G \wedge H$.

This case is almost identical to Case 3.

Case 5: $F = G \longrightarrow H$.

We use the equivalence $F \equiv \neg G \vee H$ and then we apply Cases 2 and 3.

Case 6: $F = G \longleftrightarrow H$.

We use the equivalence $F \equiv (\neg G \vee H) \wedge (\neg H \vee G)$ and then we apply Cases 2, 3, and 4.

Cases 7,8 : $F = QxG$, where $Q \in \{\forall, \exists\}$.

By IH, G has a prenex form, $G \equiv Q_1x_1 \dots Q_nx_nI$, where I has no quantifiers. Then,

$$F \equiv QxQ_1x_1 \dots Q_nx_nI$$

and the last formula is a prenex form.

Grading Criteria

1. Listing the 8 cases: 2 points
2. Case 1: 1 points
3. Case 2: 5 points
4. Case 3: 7 points
5. Cases 4,5,6 : 3 points
6. Cases 7,8: 2 points

Question 4. (20 points)

Derive \square from $S = \{\{P(x, f(x), y), P(y, z, y), Q(x, y, z)\}, \{R(f(x), g(y)), R(y, g(z)), \neg Q(x, x, z)\}, \{\neg P(g(x), y, z), Q(z, z, f(u))\}, \{\neg R(y, g(x)), \neg Q(u, u, y)\}\}$.

Do the minimal number of resolutions.

The tree is shown in Figure 1.

Grading Criteria

16.5 points for each correct step. Out of these points, 3.5 are for the resolvent, 2.5 for the mgu, and 0.5 points for the relabelings.

-2 for each extra resolvent.

Question 5 (15 points)

List $D(F, 2)$ for the Skolem form $F = \forall x \forall y ((\neg P(f(a), y) \vee \neg P(x, g(y))) \wedge (P(a, h(x)) \vee P(y, g(a)))) \wedge P(a, f(y))$.

Solution

$$D(F, 2) = \{a, f(a), g(a), h(a), f^2(a), f(g(a)), f(h(a)), g(f(a)), g^2(a), g(h(a)), h(f(a)), h(g(a)), h^2(a)\}$$

Grading Criteria

15/12 points for each right term

-1 point for each wrong term

Question 6 (20 points)

The *subsumption* is a simplification rule widely used in theorem proving.

We say that a clause $C_1 = \{L_1, \dots, L_n\}$ subsumes a clause $C_2 = \{M_1, \dots, M_m\}$ if there is a substitution $s = [z_1/t_1, \dots, z_r/t_r]$, where z_1, z_2, \dots, z_r are variables in C_1 , such that $s[C_1] \subseteq C_2$.

For example, $\{P(x, y), P(y, x)\}$ subsumes $\{P(x, x), Q(x)\}$ because

$$\{P(x, y), P(y, x)\}[y/x] = \{P(x, x), P(x, x)\} \subseteq \{P(x, x), Q(x)\}.$$

Let z_1, \dots, z_r be the variables of C_1 , and y_1, \dots, y_q be the variables of C_2 .

Prove that if C_1 subsumes C_2 then

$$\forall z_1 \dots \forall z_r C_1 \wedge \forall y_1 \dots \forall y_q C_2 \equiv \forall z_1 \dots \forall z_r C_1.$$

Proof:

Case 1: C_1 and C_2 have no variables in common.

Let \mathcal{A} be a structure with universe U and assume that \mathcal{A} is a model of $\forall z_1 \dots \forall z_r C_1$.

$$(1) \models_{\mathcal{A}} [\forall z_1 \dots \forall z_r C_1]$$

Let d_1, \dots, d_q be elements of U . We write \mathcal{A}^* for $\mathcal{A}_{[y_1 \leftarrow d_1, \dots, y_q \leftarrow d_q]}$.

Now let us evaluate $\mathcal{A}^*[s[C_1]]$.

$$\mathcal{A}^*[s[C_1]]$$

$$= \mathcal{A}^*[C_1[z_1/t_1, \dots, z_r/t_r]] \quad \text{definition of } s$$

$$= \mathcal{A}^*[C_1[z_r/t_r] \dots [z_1/t_1]] \quad t_1, \dots, t_r \text{ do not contain the variables}$$

z_1, \dots, z_r

$$= \mathcal{A}^*_{[z_1 \leftarrow \mathcal{A}^*[t_1]]}[C_1[z_r/t_r] \dots [z_2/t_2]] \quad \text{by the translation lemma}$$

$$= \mathcal{A}^*_{[z_1 \leftarrow \mathcal{A}^*[t_1], \dots, z_r \leftarrow \mathcal{A}^*[t_r]]}[C_1] \quad \text{by the translation lemma, } r - 1 \text{ times}$$

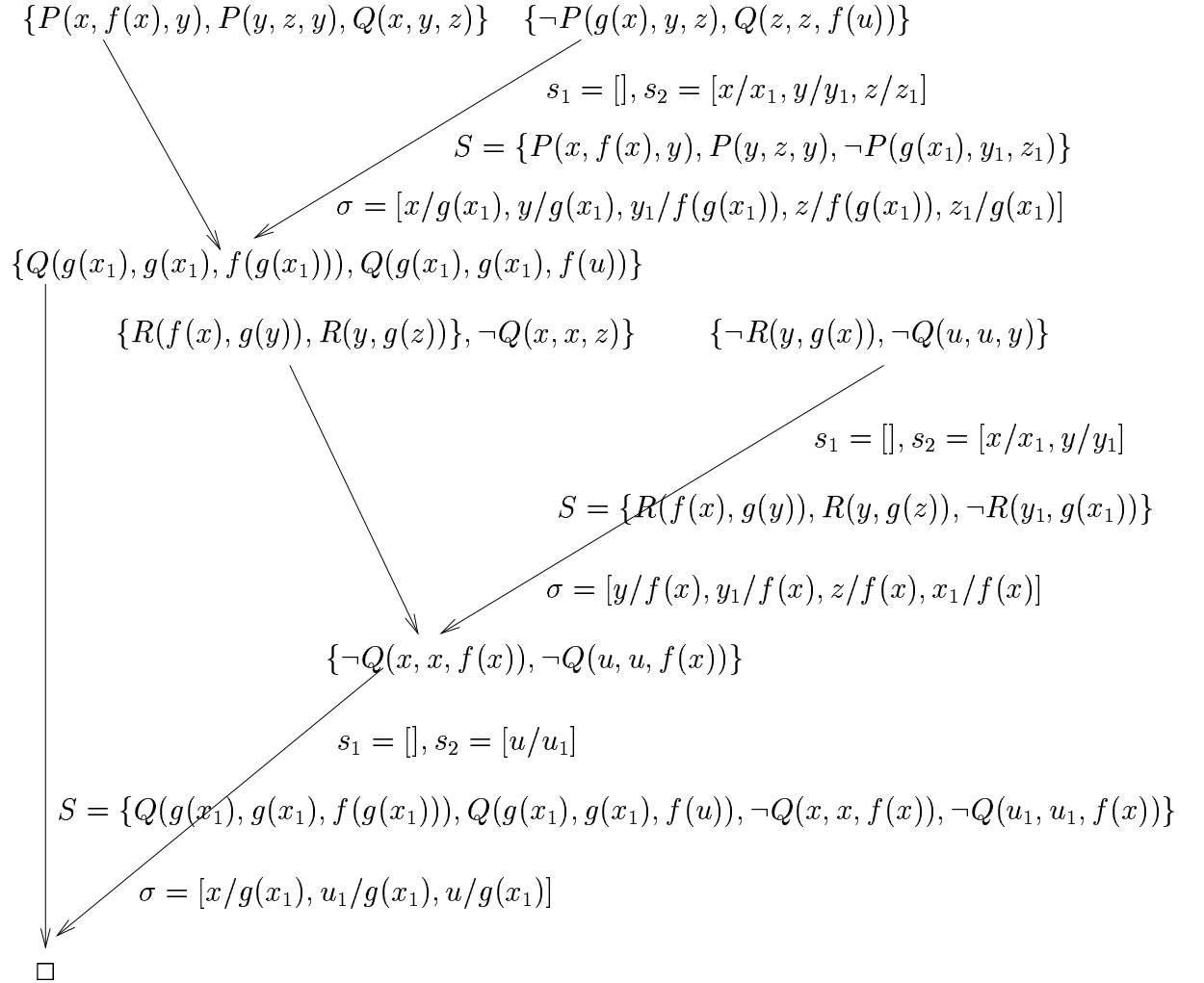


Figure 1: The tree for Question 4

$= \mathcal{A}_{[z_1 \leftarrow \mathcal{A}^*[t_1], \dots, z_r \leftarrow \mathcal{A}^*[t_r]]}[C_1]$ because y_1, \dots, y_q do not occur in C_1
 The last expression is 1 because $\mathcal{A}^*[t_1], \dots, \mathcal{A}^*[t_r]$ are elements of U , and
 (2) $\mathcal{A}_{z_1 \leftarrow e_1, \dots, z_r \leftarrow e_r}[C_1] = 1$
 for all elements $e_1, \dots, e_r \in U$.

So,

(3) $\mathcal{A}^*[s[C_1]] = 1$

Since $s[C_1] \subseteq C_2$, \mathcal{A}^* satisfies a literal of C_2 . So,

(4) $\mathcal{A}^*[C_2] = 1$

Since the assignment $\mathcal{A}^* = \mathcal{A}_{[y_1 \leftarrow d_1, \dots, y_q \leftarrow d_q]}$ was arbitrary,

(5) $\mathcal{A}[\forall y_1 \dots \forall y_q C_2] = 1$

Case 2: C_1 and C_2 have variables in common. Then let us relabel the variables of C_2 that occur in C_1 . Let $r = [y_1/u_1, \dots, y_n/u_n]$ be such a relabeling with $u_i = y_i$ when y_i is not in C_1 .

Then

(6) $r(s[C_1]) \subseteq r[C_2]$

and by the relabeling lemma

(7) $\forall y_1 \dots \forall y_q C_2 \equiv \forall u_1 \dots \forall u_q r[C_2]$.

Now the clause C_1 subsumes $r[C_2]$ and they have no variables in common.

By Case 1,

(8) $\forall z_1 \dots \forall z_r C_1 \models \forall u_1 \dots \forall u_q r[C_2]$.

Since $\forall u_1 \dots \forall u_q r[C_2]$ and $\forall y_1 \dots \forall y_q C_2$ are equivalent,

(9) $\forall z_1 \dots \forall z_r C_1 \models \forall y_1 \dots \forall y_q C_2$.

Grading Criteria

Only Case 1 was required.

1. Defining $\mathcal{A}^* = \mathcal{A}_{[y_1 \leftarrow d_1, \dots, y_q \leftarrow d_q]}$: 4 points
2. Getting $\mathcal{A}^*[s[C_1]] = \mathcal{A}_{[z_1 \leftarrow \mathcal{A}^*[t_1], \dots, z_r \leftarrow \mathcal{A}^*[t_r]]}[C_1]$: 12 points
3. Obtaining $\mathcal{A}[C_2] = 1$: 4 points.
4. Just trying: 2 points