

COT 3420
Section U2
Fall 2005

EXAM # 2 SOLUTIONS

QUESTIONS

Question 1. (10 points)

Rectify the formula

$$F = [\forall x \forall y \exists z P(x, z) \wedge \forall x \forall y \forall z (\neg P(x, y) \vee P(y, x))] \wedge P(x, x).$$

Answer: $F \equiv [\forall x_1 \exists z P(x_1, z) \wedge \forall x_2 \forall y (\neg P(x_2, y) \vee P(y, x_2))] \wedge P(x, x).$

Grading Criteria: 1. Removing the 2 redundant quantifiers : 3 points

2. Relabeling the first $\forall x$: 3 points
3. Relabeling the second $\forall x$: 4 points
4. Relabeling the free x: -3 points

Question 2 (10 points)

Skolemize the formula

$$F = \exists x_1 \forall x_2 \exists x_3 \exists x_4 \forall x_5 \exists x_6 \forall x_7 \exists x_8 F^M[a, x_1, f(x_2), x_2, x_3, x_4, x_5, x_6, x_7, x_8]$$

where F^M is the matrix of F .

Answer: $F \equiv_s \forall x_2 \forall x_5 \forall x_7 \exists x_3 \exists x_4 \exists x_6 \exists x_8 F^M[a, x_1, f(x_2), x_2, x_3, x_4, x_5, x_6, x_7, x_8][x_1/b, x_3/g(x_2), x_4/h(x_2), x_6/i(x_2, x_5), x_8/j(x_2, x_5, x_7)].$

Grading Criteria: 1. 2 points for each correct substitution.

2. -1 for each wrong function symbol, i.e. using the same symbol for different functions.

3. -1 point for each function that has the wrong arguments.

Question 3. (20 points)

Prove by structural induction that the set $S = \{\neg F \wedge G, \forall x F\}$ is adequate, i.e. every FOL formula is equivalent to an S -formula. It's Ok to reduce some cases to cases that are already solved.

Proof: By structural induction on F .

Case 1: F is an atomic formula. Then F is an S -formula.

Case 2: $F = \neg G$. By IH there is an S -formula G_1 that is equivalent to G .

Then,

$$G = \neg G$$

$$\begin{aligned} &\equiv \neg G_1 && \text{by IH} \\ &\equiv \neg G_1 \wedge E(x, x) && \text{because } E(x, x) \text{ is a tautology} \end{aligned}$$

The last formula is an S -formula being an instance of $\neg I \wedge J$ with I and J S -formulas.

Case 3: $F = G \vee H$. By IH there are S -formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$$\begin{aligned} F &= G \vee H \\ &\equiv G_1 \vee H_1 && \text{by IH} \\ &\equiv \neg \neg (G_1 \vee H_1) && \text{double negation introduction} \\ &\equiv \neg (\neg G_1 \wedge \neg H_1) && \text{De Morgan's law} \\ &\equiv \neg (\neg G_1 \wedge (\neg H_1 \wedge E(x, x))) && \text{by Case 2} \\ &\equiv \neg (\neg G_1 \wedge (\neg H_1 \wedge E(x, x))) \wedge E(x, x) && \text{by Case 2} \end{aligned}$$

The last formula is an S -formula.

Case 4: $F = G \wedge H$. By IH there are S -formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$$\begin{aligned} F &= G \wedge H \\ &\equiv G_1 \wedge H_1 && \text{by IH} \\ &\equiv \neg \neg G_1 \wedge H_1 && \text{double negation introduction} \\ &\equiv (\neg (\neg G_1 \wedge E(x, x))) \wedge H_1 && \text{by Case 2} \end{aligned}$$

The last formula is an S -formula.

Case 5: $F = G \longrightarrow H$. By IH there are S -formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$$\begin{aligned} F &= G \longrightarrow H \\ &\equiv G_1 \longrightarrow H_1 && \text{by IH} \\ &\equiv \neg G_1 \vee H_1 && \text{by } \longrightarrow \text{ elimination} \end{aligned}$$

Now we apply Case 2 to eliminate \neg and Case 3 to eliminate \vee .

Case 6: $F = G \longleftrightarrow H$. By IH there are S -formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$$\begin{aligned} F &= G \longleftrightarrow H \\ &\equiv G_1 \longleftrightarrow H_1 && \text{by IH} \\ &\equiv (G_1 \longleftrightarrow H_1) \wedge (H_1 \longleftrightarrow G_1) && \text{by } \longleftrightarrow \text{ elimination} \end{aligned}$$

Then we apply Cases 4 and 5 to eliminate \wedge and \longrightarrow .

Case 7: $F = \forall x G$. By IH there is an S -formula G_1 that is equivalent to G .

$$\begin{aligned} F &= \forall x G \\ &\equiv \forall x G_1 && \text{by IH} \end{aligned}$$

The last formula is an S -formula.

Case 8: $F = \exists x G$. By IH there is an S -formula G_1 that is equivalent to G .

$$F = \exists x G$$

x	$f^{\mathcal{A}}[x]$
3	5
4	3
5	3

 $f^{\mathcal{A}}$

$x \backslash y$	3	4	5
3	4	5	3
4	3	4	5
5	5	3	4

 $g^{\mathcal{A}}$

$x \backslash y$	3	4	5
3	0	0	0
4	1	0	1
5	0	1	0

 $P^{\mathcal{A}}$

x	$Q^{\mathcal{A}}[x]$
3	0
4	0
5	1

 $Q^{\mathcal{A}}$

Figure 1: Tables for Question 5

$\equiv \exists x G_1$ by IH
 $\equiv \neg\neg\exists x G_1$ double negation introduction
 $\equiv \neg\forall x\neg G_1$ by $\neg\exists x I \equiv \forall x\neg I$
 $\equiv \neg\forall x(\neg G_1 \wedge E(x, x))$ by Case 2
 $\equiv \neg\forall x(\neg G_1 \wedge E(x, x)) \wedge E(x, x)$ by Case 2
 The last formula is an S -formula.

- Grading Criteria:**
1. Listing the cases: 2 points.
 2. Cases 1 and 7: 1 point.
 3. Case 2: 5 points (This is key case)
 4. Case 3, 4, 8: 3 points.
 5. Cases 5,6: 1.5 points each.

Question 4. (5 points)

Close the formula $F = \exists u\forall y F^M[x, y, z, u, v, w]$ where F^M , the matrix of F , has free occurrences of x, y, z, u, v, w .

Answer: $F = \exists x\exists z\exists v\exists w\exists u\forall y F^M[x, y, z, u, v, w]$

- Grading Criteria:**
1. 1.25 points for each quantifier.
 2. -0.75 point for each insertion behind $\forall y$.

Question 5. (20 points)

The universe of \mathcal{A} is $\{3, 4, 5\}$ and the \mathcal{A} interpretations of a, x , and y are $a^{\mathcal{A}} = 3$, $x^{\mathcal{A}} = 4$ and $y^{\mathcal{A}} = 5$. The tables for the functions $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ and the predicates $P^{\mathcal{A}}$ and $Q^{\mathcal{A}}$ are displayed in Figure 1.

Evaluate the terms and the formulas below. Do not show your work, just write the answer to the right of the equal sign.

1. $\mathcal{A}[f(g(x, y))] = 3$

2. $\mathcal{A}[g(g(a, x), y)] = 4$
3. $\mathcal{A}[\neg P(y, a)] = 1$
4. $\mathcal{A}[\forall x \neg P(a, f(x))] = 1$
5. $\mathcal{A}[\exists x \forall y P(x, f(y))] = 1$
6. $\mathcal{A}[\forall x (P(y, f(x)) \vee Q(f(x)))] = 0$

Grading Criteria: 3.33 points for each right answer.

Question 6. (20 points)

Let x, u, v be 3 different variables and $\forall x F$ be a formula that does not have any occurrences of u and v .

Prove the consequence below.

$$\forall x F \models \forall u \forall v F[x/f(u, v)].$$

Proof: Let \mathcal{A} be a structure with universe U . We assume (1) and need to show (2).

- (1) $\mathcal{A}[\forall x F] = 1$
- (2) $\mathcal{A}[\forall u \forall v F[x/f(u, v)]] = 1.$

We apply the interpretation of $\forall x$ to (1) and get that (3) holds for all $d \in U$.

$$(3) \mathcal{A}_{[x \leftarrow d]}[F] = 1$$

Now, let $e_1, e_2 \in U$ and let us evaluate $\mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[F[x/f(u, v)]]$. The term $f(u, v)$ is free for x in F because u, v do not occur in F , neither as variables nor as arguments to quantifiers. So, we apply the Translation Lemma and get

$$(4) \mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[F[x/f(u, v)]] = \mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2][x \leftarrow \mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[f(u, v)]]}[F]$$

We evaluate $\mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[f(u, v)]$.

$$(5) \mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[f(u, v)] = f^{\mathcal{A}}[e_1, e_2]$$

We use (5) to replace $\mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[f(u, v)]$ in (4) and get (6).

$$(6) \mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[F[x/f(u, v)]] = \mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2][x \leftarrow f^{\mathcal{A}}[e_1, e_2]]}[F]$$

The variables u and v do not occur in F , so the structures $\mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2][x \leftarrow f^{\mathcal{A}}[e_1, e_2]]}$ and $\mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[e_1, e_2]]}$ agree on F . By The Agreement Theorem,

$$(7) \mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2][x \leftarrow f^{\mathcal{A}}[e_1, e_2]]}[F] = \mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[e_1, e_2]]}[F].$$

We use (7) to replace $\mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2][x \leftarrow f^{\mathcal{A}}[e_1, e_2]]}[F]$ in (6).

$$(8) \mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[F[x/f(u, v)]] = \mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[e_1, e_2]]}[F]$$

Now we apply (3) and get that,

$$(9) \mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[e_1, e_2]]}[F] = 1.$$

From (8) and (9) we obtain

$$(10) \mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[F[x/f(u, v)]] = 1.$$

Now,

$\mathcal{A}[\forall u \forall v F[x/f(u, v)]] = 1$
 iff for all $e_1 \in U$, $\mathcal{A}_{[u \leftarrow e_1]}[\forall v F[x/f(u, v)]] = 1$ by the interpretation of $\forall u$
 iff for all $e_1 \in U$, for all $e_2 \in U$, $\mathcal{A}_{[u \leftarrow e_1][v \leftarrow e_2]}[F[x/f(u, v)]] = 1$ by the interpretation of $\forall v$
 iff for all $e_1 \in U$, for all $e_2 \in U$, true by (10)
 iff true.
 So, $\mathcal{A}[\forall u \forall v F[x/f(u, v)]] = 1$ **Q.E.D.**

Grading Criteria: 1. Writing down the assumption (1) and the goal (2) : 3 points

2. Writing (3): 2 points.
3. Deriving (10) 13 points.
4. The wrap up : 2 points.