COT 3420
Section U2
Fall 2005

FINAL EXAM ANSWERS

QUESTIONS
Question 1. (15 points)

Answers: 1. a 2. c 3. d 4. a 5. d 6. a 7. a 8. c 9. b
10. c

Grading Criteria 1.5 points for each correct answer.

Question 2. (15 points)
The answer is in Figure 1.
Question 3. (15 points)

Answer: $D(F,2) = \{a, f(a,a), f(a, f(a,a)), f(f(a,a),a), f(f(a,a), f(a,a))\}$.
So, $E(F,2) = \{F^M[x/a, y/a], F^M[x/a, y/f(a,a)], F^M[x/a, y/f(a, f(a,a))],$
$F^M[x/a, y/f(f(a,a),a)], F^M[x/a, y/f(f(a,a), f(a,a))], F^M[x/f(a,a), y/a],$
$F^M[x/f(a,a), y/f(a,a)], F^M[x/f(a,a), y/f(a, f(a,a))], F^M[x/f(a,a), y/f(f(a,a),a)],$
$F^M[x/f(a,a), y/f(f(a,a), f(a,a))], F^M[x/f(a, f(a,a)), y/a], F^M[x/f(a, f(a,a)), y/f(a,a)],$
$F^M[x/f(a, f(a,a)), y/f(a, f(a,a))], F^M[x/f(a, f(a,a)), y/f(f(a,a),a)],$
$F^M[x/f(a, f(a,a)), y/f(f(a,a), f(a,a))], F^M[x/f(f(a,a),a), y/a],$
$F^M[x/f(f(a,a),a), y/f(a,a)], F^M[x/f(f(a,a),a), y/f(a, f(a,a))],$
$F^M[x/f(a,a),a), y/f(f(a,a),a)], F^M[x/f(f(a,a),a), y/f(f(a,a), f(a,a))],$
$F^M[x/f(f(a,a), f(a,a)), y/a], F^M[x/f(f(a,a), f(a,a)), y/f(a,a)],$
$F^M[x/f(f(a,a), f(a,a)), y/f(a, f(a,a))], F^M[x/f(f(a,a), f(a,a)), y/f(f(a,a),a)],$
$F^M[x/f(f(a,a), f(a,a)), y/f(f(a,a), f(a,a))]\}$
Grading Criteria: 3/5 points for each correct proposition; -3/5 for each incorrect one.

Question 4 (20 points)
a. Let σ be a mgu of S. Since $S_1 \subseteq S$, σ is a unifier of S_1. So, S_1 is unifiable. The same argument works for S_2.

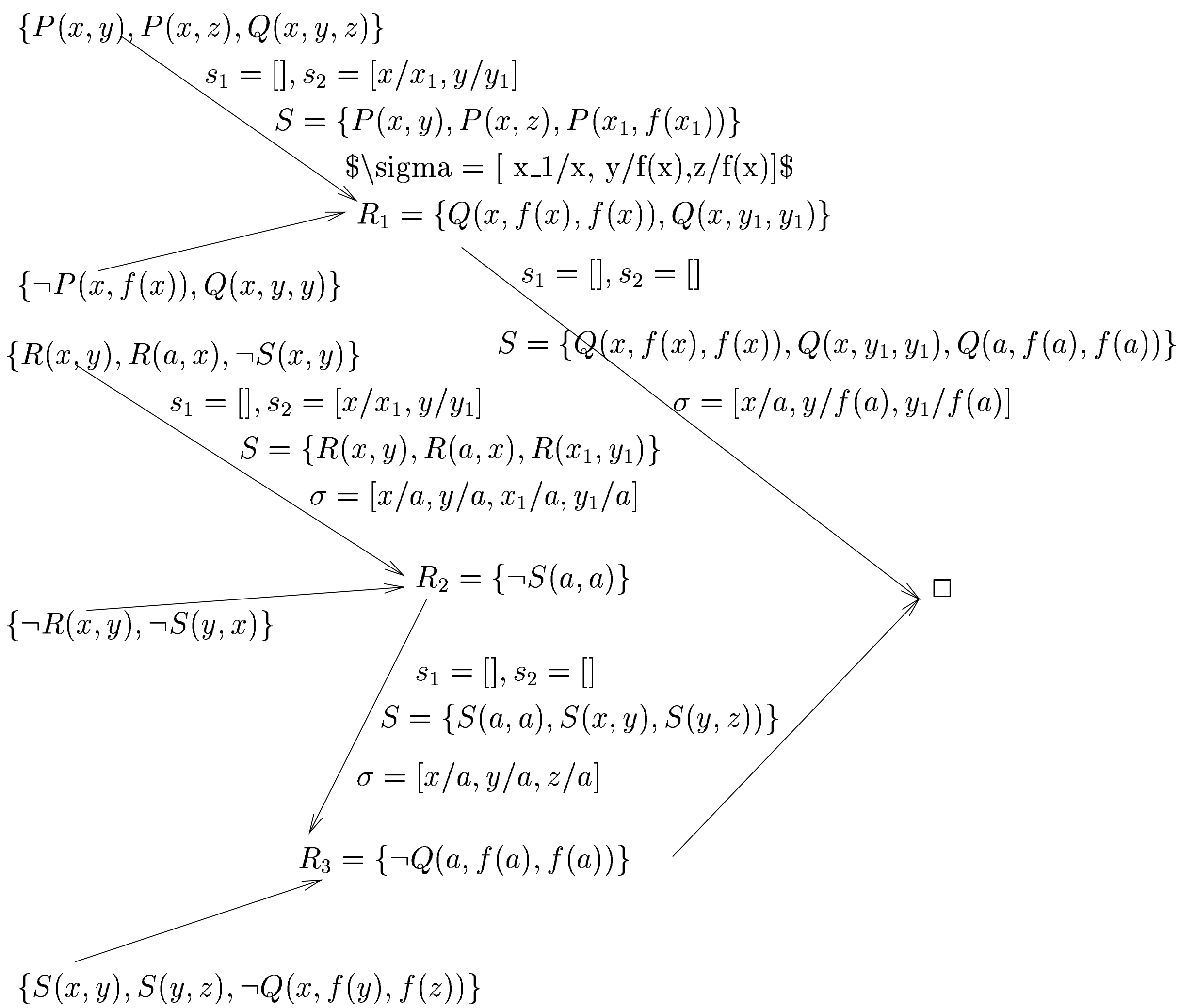

Figure 1: The Answer to Q2

b. Let σ be a mgu of S and s_1 a mgu of S_1. Let $s_1 = [x_1/t_1, \ldots x_n/t_n]$. Since σ is a unifier of S_1 there is substitution π such that $\sigma = \pi \diamond s_1$. Now let α be the restriction of π to the variables of C_1 that not in $\{x_1, \ldots, x_n\}$, i.e. $\alpha = \pi \uparrow ((Var[C_1] - \{x_1, \ldots x_n\}) \cap dom(\pi))$.

We will prove the following fact.

Fact For all variables x in C_1, $\sigma[x] = \bar{\alpha}(s_1[x])$.

Proof: By cases.

Case 1: $x = x_i$. Then the variables of t_i are in the set $Var[C_1] - \{x_1, \ldots, x_n\}$. So, $\bar{\pi}[t_i] = \bar{\alpha}[t_i]$.

$\sigma[x_i] = \bar{\pi}[s_1[x_i]]$ because $\sigma = \pi \diamond s_1$

$= \bar{\pi}[t_i]$ since $s_1[x_i] = t_i$

$= \bar{\alpha}[t_i]$ by the above equality

$= \bar{\alpha}[s_1[x_i]]$

Case 2: $x \notin \{x_1, \ldots x_n\}$. Then $s_1[x] = x$, and $\pi[x] = \alpha[x]$.

$\sigma[x] = \bar{\pi}[s_1[x]]$ because $\sigma = \pi \diamond s_1$

$= \bar{\pi}[x]$ since $s_1[x] = x$

$= \bar{\alpha}[x]$ definition of α

$= \bar{\alpha}[s_1[x]]$ because $s_1[x] = x$

Q.E.D. Fact

In a similar way, we show that there is substitution β such that for all $y \in C_2$, $\sigma[y] = \bar{\beta}(s_2[y])$.

Now, the domain of α contains only variables from C_1 and the domain of β only variables of C_2. So, $\alpha \cup \beta$ is a substitution and we have the equalities (1) and (2).

(1) $(\alpha \cup \beta)[s_1[A_1]] = \alpha[s_1[A_1]]$

(2) $(\alpha \cup \beta)[s_2[B_1]] = \beta[s_2[B_1]]$

From (1) and (2) we get that $\alpha \cup \beta$ is a unifier of $\{s_1[A], s_2[B]\}$.

c. Let s be a mgu of $\{s_1[A], s_2[B]\}$. We showed in part b that $\alpha \cup \beta$ is a unifier of $\{s_1[A], s_2[B]\}$. So, $\alpha \cup \beta$ factors through s, i.e. there is a substitution π such that $\alpha \cup \beta = \pi \diamond s$. Since $\sigma[z] = (\alpha \cup \beta)[z]$ for $z \in Var[C_1] \cap Var[C_2]$ and σ is a mgu of S, $\sigma = \alpha \cup \beta$. **Q.E.D.**

Question 5. (20 points)

Proof: If S has a finite unsatisfiable subset, then S is unsatisfiable. We need to show that whenever S is unsatisfiable, S has a finite unsatisfiable subset. Let $S = \{F_0, F_1, \ldots, F_n, \ldots\}$ be an unsatisfiable set of FOL formulas. The algorithm for finding an unsatisfiable subset is shown below.

1. Skolemize every formula in S, getting a set of clauses (with variables) U.

2. Find the Herbrand Expansion of $E[U]$.

3. Apply the compactness therem for propositional logic to find a finite unsatisfiable subset $V \subseteq E[U]$.

4. Use the lifting lemma to get a finite unsatisfiable subset $T \subseteq U$.

Let us show the correctness of the algorithm. Since S is unsatisfiable, U is unsatisfiable by Skolem's Theorem. Then $E[U]$ is unsatisfiable by Herbrand's Theorem. The correctness of steps 3 and 4 are guaranteed by the compactness theorem for propositional logic and the lifting lemma, in this order.

There is a little glitch in this algorithm. The Skolemization process requires new function symbols and S may use all symbols. For this reason, we do not apply the algorithm to S, but to R, the set of formulas obtained from S by replacing every function symbol f_i^n by f_{2i}^n. This way, we are free to use functions with odd indices for skolemization.

How do we find an unsatisfiable subset of S? First we Skolemize every formula G_i of R and get $G_i \equiv_s \{C_{i,0}, \ldots C_{i,n_i}\}$, where $C_{i,j}$ are clauses. Then, the compactness theorem for propositional logic generates an unsatisfiable finite set of ground clauses $V = \{C'_{i_0,j_0}, \ldots, C'_{i_k,j_k}\}$. The lifting lemma will produce the unsatisfiable set $T = \{C_{i_0,j_0}, \ldots, C_{i_k,j_k}\}$. Since the clause $C_{i,j}$ was generated by formula G_i of R, $W = \{G_{i_0}, \ldots, G_{i_k}\}$ is unsatisfiable. Now we convert the clauses G_i to F_i by changing every f_{2l}^n to f_l^n. So, we get a finite unsatisfiable subset $X = \{F_{i_0}, \ldots, F_{i_k}\}$ of S. **Q.E.D.**

Question 6. (20 points)

We create the data base shown below.

```
% create the arcs
arc(a,b).
arc(b,a).
arc(a,c).
path1(X,X).
path1(X,Y) :- arc(X,Z), path1(Z,Y).
path2(X,X). path1(X,X).
path1(X,Y) :- arc(X,Z), path1(Z,Y).
path2(X,X).
path2(X,Y):- path2(X,Z), arc(Z,Y).
```

What will be printed out by the queries below? Write yes, no, or stack overflow next to the query.

?- path1(a,c).	stack overflow
?- path2(a,c).	yes
?- path1(c,b).	no
?- path2(c,b).	stack overflow

Grading Criteria 5 points for each correct answer