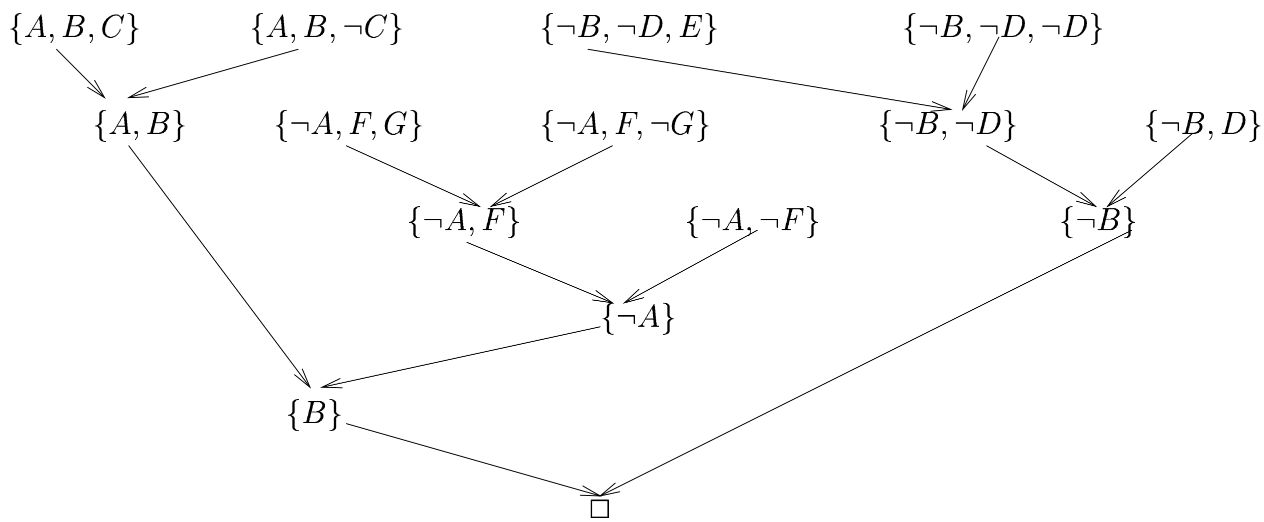

Figure 1: The answer to Q2

COT 3420
Spring 2006

EXAM # 2 ANSWERS

Question 1. (39 points)
1. b 2. b 3. a 4. d 5. b 6. c 7. a 8. a 9. d 10. c 11. a 12. c 13. c
Grading Criteria: 3 points for each correct answer.

Question 2. (20 points)
The resolution tree is hown in Figure 1.
Grading Criteria: 1. 20/7 points for each correct derivation step that leads to □
2. -4 points for each incorrect derivation step
3. -1 point for each missing or misdirected arrow
4. -2 points for finding □ in 9 or more steps

Question 3. (25 points)

Prove the generalized distributivity law,

$F \lor \bigwedge_{i=1}^{n} G_i \equiv \bigwedge_{i=1}^{n}(F \lor G_i)$.

Write your proof on a blank sheet of paper.

Hint: Use induction on n.

Proof: We recall that:

$\bigwedge_{i=m}^{n} G_i = \mathbf{T}$ if $m > n$ (rule 1)

$\bigwedge_{i=m}^{n} G_i = G_n$ if $m = n$ (rule 2)

$\bigwedge_{i=m}^{n} G_i = (\bigwedge_{i=m}^{n-1} G \land G_n)$ if $m < n$ (rule 3)

Now let's construct the proof.

Case $n = 0$.

LHS $= F \lor \bigwedge_{i=1}^{n} G_i = F \lor \bigwedge_{i=1}^{0} G_i$

$= F \lor \mathbf{T}$ rule 1

$\equiv \mathbf{T}$ tautology law

RHS $= \bigwedge_{i=1}^{n}(F \lor G_i) = \bigwedge_{i=1}^{0}(F \lor G_i)$

$= \mathbf{T}$ rule 1

So, both sides are equivalent.

Case $n = 1$.

LHS $= F \lor \bigwedge_{i=1}^{n} G_i = F \lor \bigwedge_{i=1}^{1} G_i$

$= F \lor G_1$ rule 2

RHS $= \bigwedge_{i=1}^{n}(F \lor G_i) = \bigwedge_{i=1}^{1}(F \lor G_i)$

$= F \lor G_1$ rule 2

So, both sides are equal.

Case $n = 2$.

LHS $= F \lor \bigwedge_{i=1}^{n} G_i = F \lor \bigwedge_{i=1}^{2} G_i$

$= F \lor (\bigwedge_{i=1}^{1} G_i \land G_2)$ rule 3

$= F \lor (G_1 \land G_2)$ rule 2

RHS $= \bigwedge_{i=1}^{n}(F \lor G_i) = \bigwedge_{i=1}^{2}(F \lor G_i)$

$= (\bigwedge_{i=1}^{1}(F \lor G_i) \land (F \lor G_2))$ rule 3

$= ((F \lor G_1) \land (F \lor G_2))$ rule 2

We prove that LHS $\equiv$RHS by truth tables.

F	G_1	G_2	$F \vee G_1$	$F \vee G_2$	$(F \vee G_1) \wedge (F \vee G_2)$	$(G_1 \wedge G_2)$	$F \vee (G_1 \wedge G_2)$
0	0	0	0	0	0	0	0
0	0	1	0	1	0	0	0
0	1	0	1	0	0	0	0
0	1	1	1	1	1	1	1
1	0	0	1	1	1	0	1
1	0	1	1	1	1	0	1
1	1	0	1	1	1	0	1
1	1	1	1	1	1	1	1

The columns under $(F \vee G_1) \wedge (F \vee G_2)$ and $F \vee (G_1 \wedge G_2)$ are identical, so they are equivalent. This is the distributivity equivalence.

Inductive step ($n \geq 2$).

Assume (IH).

(IH) $F \vee \bigwedge_{i=1}^{n} G_i \equiv \bigwedge_{i=1}^{n}(F \vee G_i)$

We need to show

$F \vee \bigwedge_{i=1}^{n+1} G_i \equiv \bigwedge_{i=1}^{n+1}(F \vee G_i)$

LHS $= F \vee \bigwedge_{i=1}^{n+1} G_i$

$= F \vee (\bigwedge_{i=1}^{n} G_i \wedge G_{n+1})$ rule 3, because $n+1 > 1$

$\equiv (F \vee \bigwedge_{i=1}^{n} G_i) \wedge (F \vee G_{n+1})$ Case $n = 2$

$\equiv \bigwedge_{i=1}^{n}(F \vee G_i) \wedge (F \vee G_{n+1})$ by IH

$= \bigwedge_{i=1}^{n+1}(F \vee G_i)$ rule 3

$=$ RHS. **Q.E.D.**

Grading Criteria: Case $n = 0$: 4 points; 1 point for the reasons

Case $n = 1$: 4 points; 1 point for the reasons

Case $n = 2$: 9 points; the truth table is worth 5 points; 1.5 points for the resons

The inductive case: 8 points; the IH 2 points; the derivation 4.5 points; the explanations 1.5 points

Question 4. (25 points)

Show, by structural induction, that the set $S = \{F \wedge \neg G, \mathbf{T}\}$ is adequate where $\mathbf{T}$ is a tautology.

Let $\diamond$ be defined as $F \diamond G = F \wedge \neg G$. We need to show that $S = \{\diamond, \mathbf{T}\}$ is adequate. First, we prove the lemma below.

Lemma: 1. $\neg F \equiv \mathbf{T} \diamond F$

2. $F \vee G \equiv \mathbf{T} \diamond ((\mathbf{T} \diamond F) \diamond G)$

3. $F \wedge G \equiv F \diamond (\mathbf{T} \diamond G)$

4. $F \longrightarrow G \equiv \mathbf{T} \diamond (F \diamond G)$
5. $F \longleftrightarrow G \equiv (\mathbf{T} \diamond (F \diamond G)) \diamond (G \diamond F)$

Proof of the lemma:

1. $\neg F \equiv \mathbf{T} \wedge \neg F$ tautology law
$= \mathbf{T} \diamond F$ definition of $\diamond$
2. $F \vee G \equiv \neg\neg(F \vee G)$ double $\neg$ introduction
$\equiv \neg(\neg F \wedge \neg G)$ DeMorgan's law
$\equiv \neg(\neg F \diamond G)$ definition of $\diamond$
$\equiv \neg((\mathbf{T} \diamond F) \diamond G)$ part 1
$\equiv \mathbf{T} \diamond ((\mathbf{T} \diamond F) \diamond G)$ part 1
3. $F \wedge G \equiv F \wedge \neg\neg G$ double $\neg$ introduction
$\equiv F \diamond \neg G$ definition of $\diamond$
$\equiv F \diamond (\mathbf{T} \diamond G)$ part 1
4. $F \longrightarrow G \equiv \neg F \vee G$ $\longrightarrow$-elim
$\equiv \neg\neg(\neg F \vee G)$ double $\neg$ introduction
$\equiv \neg(\neg\neg F \wedge \neg G)$ DeMorgan's law
$\equiv \neg(F \wedge \neg G)$ double $\neg$ elim
$\equiv \neg(F \diamond G)$ definition of $\diamond$
$\equiv \mathbf{T} \diamond (F \diamond G)$ part 1
5. $F \longleftrightarrow G \equiv (F \longrightarrow G) \wedge (G \longrightarrow F)$ $\longleftrightarrow$-elim
$\equiv (\neg F \vee G) \wedge (\neg G \vee F)$ $\longrightarrow$-elim
$\equiv (\neg F \vee \neg\neg G) \wedge (\neg G \vee \neg\neg F)$ double $\neg$ introduction twice
$\equiv \neg(F \wedge \neg G) \wedge \neg(G \wedge \neg F)$ DeMorgan's law twice
$\equiv \neg(F \diamond G) \wedge \neg(G \diamond F)$ definition of $\diamond$
$\equiv \neg(F \diamond G) \diamond (G \diamond F)$ definition of $\diamond$
$\equiv (\mathbf{T} \diamond (F \diamond G)) \diamond (G \diamond F)$ part 1

Q.E.D. lemma

Now, let us prove that $S == \{\diamond, \mathbf{T}\}$ is adequate by structural induction on F.

Proof:

Case 1: F is an atom. Then F is an S-formula.

Case 2: $F = \neg G$. By IH there is an S-formula G_1 such that $G \equiv G_1$. Then
$F = \neg G$
$\equiv \neg G_1$ by IH
$\equiv \mathbf{T} \diamond G_1$ by Lemma, part 1
The last formula is an S-formula.

Case 3: $F = G \vee H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$. Then

$F = G \vee H$

$\equiv G_1 \vee H_1$ by IH

$\equiv \mathbf{T} \diamond ((\mathbf{T} \diamond G_1) \diamond H_1)$ Lemma, part 2

The last formula is S.

The rest of the cases ($F = G \wedge H, F = G \longrightarrow H, F = G \longleftrightarrow H$ are solved in a similar fashion, using the corresponding parts of the lemma. **Q.E.D.**

Grading Criteria: 1. Listing the 6 cases: 3 points

2. Case 1: 1 point

3. Case 2: 5 points (the corresponding part of the lemma : 3.5 points, the IH 1, the derivation 0.5 points)

4. Case 3: 4 points (the corresponding part of the lemma : 3 points, the IH 0.5, the derivation 0.5 points)

5. Case 4: 4 points

6. Case 5: 4 points

7. Case 6: 4 points

Question 5. (20 points)

Find a CNF for the formula

$F = [\neg(A \wedge (B \vee C)) \longleftrightarrow (\neg B \vee \neg(C \vee D))]$.

Solution

$F = [\neg(A \wedge (B \vee C)) \longleftrightarrow (\neg B \vee \neg(C \vee D))]$ line 1

$\equiv [\neg(A \wedge (B \vee C)) \longrightarrow (\neg B \vee \neg(C \vee D))] \wedge [(\neg B \vee \neg(C \vee D)) \longrightarrow \neg(A \wedge (B \vee C))]$ $\longleftrightarrow$-elim , line 2

$\equiv [\neg\neg(A \wedge (B \vee C)) \vee (\neg B \vee \neg(C \vee D))] \wedge [\neg(\neg B \vee \neg(C \vee D)) \vee \neg(A \wedge (B \vee C))]$ $\longrightarrow$-elim , line 3

$\equiv [(A \wedge (B \vee C)) \vee (\neg B \vee (\neg C \wedge \neg D))] \wedge [(\neg\neg B \wedge \neg\neg(C \vee D)) \vee (\neg A \vee \neg(B \vee C))]$ double $\neg$-elim, De Morgan's 3 times, line 4

$\equiv [(A \wedge (B \vee C)) \vee (\neg B \vee (\neg C \wedge \neg D))] \wedge [(B \wedge (C \vee D)) \vee (\neg A \vee (\neg B \wedge \neg C))]$ double $\neg$-elim twice, De Morgan's, line 5

$\equiv [(A \wedge (B \vee C)) \vee ((\neg B \vee \neg C) \wedge (\neg B \vee \neg D))] \wedge [(B \wedge (C \vee D)) \vee ((\neg A \vee \neg B) \wedge (\neg A \vee \neg C))]$ distributivity twice, line 6

$\equiv (A \vee \neg B \vee \neg C) \wedge (A \vee \neg B \vee \neg D) \wedge (B \vee C \vee \neg B \vee \neg C) \wedge (B \vee C \vee \neg B \vee \neg D) \wedge (B \vee \neg A \vee \neg B) \wedge (B \vee \neg A \vee \neg C) \wedge (C \vee D \vee \neg A \vee \neg B) \wedge (C \vee D \vee \neg A \vee \neg C)$ generalized distributivity, line 7

$\equiv (A \vee \neg B \vee \neg C) \wedge (A \vee \neg B \vee \neg D) \wedge (\neg A \vee B \vee \neg C) \wedge (\neg A \vee \neg B \vee C \vee D)$ tautology elim, ordering the literals, line 8

Grading Criteria:

You get credit up to the first line where you made an error.
If the error occured on line 2 you get 1 point.
If the error occured on line 3 you get 2 points.
If the error occured on line 4 you get 4 points.
If the error occured on line 5 you get 7 points.
If the error occured on line 6 you get 10 points.
If the error occured on line 7 you get 13 points.
If the error occured on line 8 you get 16 points.
If there is no error 19 points.
1 point is for reasons.