COT 3420
SPRING 2006

EXAM # 3 ANSWERS

QUESTIONS

Question 1. (30 points)

The universe of structure $\mathcal{A}$ is $\{3, 4, 5\}$. The $\mathcal{A}$ interpretations of a, x, and y are $a^{\mathcal{A}} = 4$, $x^{\mathcal{A}} = 5$, $y^{\mathcal{A}} = 3$. The tables for the functions $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ and the predicates $P^{\mathcal{A}}$ and $Q^{\mathcal{A}}$ are shown in Figure 1. Evaluate the terms and formulas below. Do not show your work, just write the answer after the equal sign.

1. $\mathcal{A}[f(x)] = f^{\mathcal{A}}[x^{\mathcal{A}}] = f^{\mathcal{A}}[5] = 5$
2. $\mathcal{A}[g(x,x)] = g^{\mathcal{A}}[x^{\mathcal{A}}, x^{\mathcal{A}}] = g^{\mathcal{A}}[5,5] = 3$
3. $\mathcal{A}[g(g(a,y), f(x))] = g^{\mathcal{A}}[g^{\mathcal{A}}[a^{\mathcal{A}}, y^{\mathcal{A}}], f^{\{}[x^{\mathcal{A}}]] =$
$g^{\mathcal{A}}[g^{\mathcal{A}}[4,3], f^{\mathcal{A}}[5]] = g^{\mathcal{A}}[3,5] = 4$
4. $\mathcal{A}[Q(g(y,x))] = Q^{\mathcal{A}}[g^{\mathcal{A}}[y^{\mathcal{A}}, x^{\mathcal{A}}]] = Q^{\mathcal{A}}[g^{\mathcal{A}}[3,5] = Q^{\mathcal{A}}[4] = 1$
5. $\mathcal{A}[P(x, f(a))] = P^{\mathcal{A}}[x^{\mathcal{A}}, f^{\mathcal{A}}[a^{\mathcal{A}}]] = P^{\mathcal{A}}[5, f^{\mathcal{A}}[4]] = P^{\mathcal{A}}[5,4] = 0$
6. $\mathcal{A}[E(f(x), g(y,a))] = 0$ because $\mathcal{A}[f(x)] \neq \mathcal{A}[g(y,a)]$
$\mathcal{A}[f(x)] = f^{\mathcal{A}}[x^{\mathcal{A}}] = f^{\mathcal{A}}[5] = 5$
$\mathcal{A}[g(y,a)] = g^{\mathcal{A}}[y^{\mathcal{A}}, a^{\mathcal{A}}] = g^{\mathcal{A}}[3,4] = 3$
7. $\mathcal{A}[\forall x P(f(x), x)] = P^{\mathcal{A}}[5,3] \boxed{\wedge} P^{\mathcal{A}}[4,4] \boxed{\wedge} P^{\mathcal{A}}[5,5] = 1$
8. $\mathcal{A}[\forall x Q(f(x))] = Q^{\mathcal{A}}[5] \boxed{\wedge} Q^{\mathcal{A}}[4] \boxed{\wedge} Q^{\mathcal{A}}[5] = 0$
9. $\mathcal{A}[\exists x \forall y \neg P(x,y)] = \mathcal{A}_{[x\leftarrow 3]}[\forall y \neg P(x,y)] \boxed{\vee}\, \mathcal{A}_{[x\leftarrow 4]}[\forall y \neg P(x,y)] \boxed{\vee} \mathcal{A}_{[x\leftarrow 5]}[\forall y \neg P(x,y)]$
$= [\boxed{\neg} P^{\mathcal{A}}[3,3] \boxed{\wedge} \boxed{\neg} P^{\mathcal{A}}[3,4] \boxed{\wedge} \boxed{\neg} P^{\mathcal{A}}[3,5]] \boxed{\vee}\, \mathcal{A}_{[x\leftarrow 4]}[\forall y \neg P(x,y)] \boxed{\vee} \mathcal{A}_{[x\leftarrow 5]}[\forall y \neg P(x,y)] =$
1
10. $\mathcal{A}[\forall x \exists y P(y,x)] = \mathcal{A}_{[x\leftarrow 3]}[\exists y P(y,x)] \boxed{\wedge}\, \mathcal{A}_{[x\leftarrow 4]}[\exists y P(y,x)] \boxed{\wedge} \mathcal{A}_{[x\leftarrow 5]}[\exists y P(x,y)]$
$= [P^{\mathcal{A}}[3,3] \boxed{\vee} P^{\mathcal{A}}[4,3] \boxed{\vee} P^{\mathcal{A}}[5,3]] \boxed{\wedge}\, [P^{\mathcal{A}}[3,4] \boxed{\vee} P^{\mathcal{A}}[4,4] \boxed{\vee} P^{\mathcal{A}}[5,4]] \boxed{\wedge}$
$[P^{\mathcal{A}}[3,5] \boxed{\vee} P^{\mathcal{A}}[4,5] \boxed{\vee} P^{\mathcal{A}}[5,5]] = 1$

Question 2. (25 points)

Prove that if x is not free in G, then $\exists x F \longrightarrow G \equiv \forall x(F \longrightarrow G)$.

Syntactic Proof:

$\exists x F \longrightarrow G \equiv \neg \exists x F \vee G$ $\quad$ $\longrightarrow$-elim
$\equiv \forall x \neg F \vee G$ $\quad$ because $\neg \exists F \equiv \forall x \neg F$

x	$f^{\mathcal{A}}[x]$
3	5
4	4
5	5

$f^{\mathcal{A}}$

$x \backslash y$	3	4	5
3	5	3	4
4	3	4	5
5	4	5	3

$g^{\mathcal{A}}$

$x \backslash y$	3	4	5
3	0	0	0
4	0	1	1
5	1	0	1

$P^{\mathcal{A}}$

x	$Q^{\mathcal{A}}[x]$
3	1
4	1
5	0

$Q^{\mathcal{A}}$

Figure 1: Tables for Question 1

$\equiv \forall x(\neg F \vee G)$ since x is not free in G
$\equiv \forall x(F \longrightarrow G)$ by $\longrightarrow$-intro

Semantic Proof:

Let $\mathcal{A}$ be a structure with universe U. We show that $\mathcal{A}[\exists x F \longrightarrow G] = 1$ iff $\mathcal{A}[\forall x(F \longrightarrow G)] = 1$

$\mathcal{A}[\exists x F \longrightarrow G] = 1$
iff $\mathcal{A}[\exists x F] = 0$ or $\mathcal{A}[G] = 1$ interpretation of $\longrightarrow$
iff for all $d \in U$ $\mathcal{A}_{[x\leftarrow d]}[F] = 0$, or $\mathcal{A}[G] = 1$ interpretation of $\exists x$
iff for all $d \in U$, $\mathcal{A}_{[x\leftarrow d]}[F] = 0$ or $\mathcal{A}[G] = 1$
iff for all $d \in U$, $\mathcal{A}_{[x\leftarrow d]}[F] = 0$ or $\mathcal{A}_{[x\leftarrow d]}[G] = 1$ see remark
iff for all $d \in U$, $\mathcal{A}_{[x\leftarrow d]}[F \longrightarrow G] = 1$ interpretation of $\longrightarrow$
iff $\mathcal{A}[\forall x(F \longrightarrow G)] = 1$. interpretation of $\forall x$

Remark: Since x is not free in G, $\mathcal{A}$ and $\mathcal{A}_{[x\leftarrow d]}$ agree on G. By the Agreement Theorem, $\mathcal{A}[G] = \mathcal{A}_{[x\leftarrow d]}[G]$.

Grading Criteria:

1. Error in the first line like $\exists x F \longrightarrow G \equiv \exists x(F \longrightarrow G)$ or $\exists x F \longrightarrow G \equiv existsx \neg F \longrightarrow G$: -15 points.
2. Error in the first line due to the wrong interpretation of $\longrightarrow$ or interpreting $\exists$ before $\longrightarrow$: -15 points
3. Not using the fact that x is not free in G or not using the consequence $\mathcal{A}_{[x\leftarrow d]}$ and $\mathcal{A}$ agree on G: -5 points
4. Confusing the interpretations of $\exists x$ and $\forall x$: -10 points
5. Not giving reasons for iff's: -5 points.
6. Skipping steps: -3 to -5 points.

Question 3. (25 points)

We define the set of connectives $S = \{F \longrightarrow G, \Box, \exists x F\}$ where x can be any variable. Prove, by structural induction on F, that S is adequate for FOL.

Proof:

Lemma: 1. $\neg F \equiv F \longrightarrow \Box$.

2. $F \vee G \equiv (F \longrightarrow \Box) \longrightarrow G$

3. $F \wedge G \equiv (F \longrightarrow (G \longrightarrow \Box)) \longrightarrow \Box$

4. $F \longleftrightarrow G \equiv ((F \longrightarrow G) \longrightarrow ((G \longrightarrow F) \longrightarrow \Box)) \longrightarrow \Box$

5. $\forall x F \equiv \exists x (F \longrightarrow \Box) \longrightarrow \Box$

Proof: 1. $\neg F \equiv \neg F \vee \Box$ tautology law

$\equiv F \longrightarrow \Box$ $\longrightarrow$ intro

2. $F \vee G \equiv \neg\neg F \vee G$ double negation introduction

$\equiv (\neg F \longrightarrow G)$ $\longrightarrow$ intro

$\equiv (F \longrightarrow \Box) \longrightarrow G$ part 1

3. $F \wedge G \equiv \neg\neg(F \wedge G)$ double negation introduction

$\equiv \neg(\neg F \vee \neg G)$ DeMorgan's law

$\equiv \neg(F \longrightarrow \neg G)$ $\longrightarrow$ intro

$\equiv \neg(F \longrightarrow (G \longrightarrow \Box))$ part 1

$\equiv (F \longrightarrow (G \longrightarrow \Box)) \longrightarrow \Box$ part 1

4. $F \longleftrightarrow G \equiv (F \longrightarrow G) \wedge (G \longrightarrow F)$ $\longleftrightarrow$ -elim

$\equiv ((F \longrightarrow G) \longrightarrow ((G \longrightarrow F) \longrightarrow \Box)) \longrightarrow \Box$ part 3

5. $\forall x F \equiv \neg\neg\forall x F$ double negation introduction

$\neg\exists x \neg F$ pushing $\neg$ past $\forall$

$\equiv \exists x \neg F \longrightarrow \Box$ part 1

$\equiv \exists x (F \longrightarrow \Box) \longrightarrow \Box$ part 1

Q.E.D lemma

Here is the proof that S is adequate, i.e. that every formula F has an equivalent S-formula.

Case 1: F is an atomic formula. Then F is an S-formula.

Case 2: $F = \neg G$. By IH there is an S-formula G_1 such that $G \equiv G_1$.

$F = \neg G \equiv \neg G_1$ by IH

$\equiv G_1 \longrightarrow \Box$ part 1 of the lemma

The last formula is an S-formula.

Case 3: $F = G \vee H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$F = G \vee H \equiv G_1 \vee H_1$ by IH

$\equiv (G_1 \longrightarrow \Box) \longrightarrow H_1$ part 2 of the lemma

The last formula is S.

Case 4: $F = G \wedge H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$F = G \wedge H \equiv G_1 \wedge H_1$ by IH

$\equiv (G_1 \longrightarrow (H_1 \longrightarrow \Box)) \longrightarrow \Box$ part 3 of the lemma

The last formula is S.

Case 5: $F = G \longrightarrow H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$F = G \longrightarrow H \equiv G_1 \longrightarrow H_1$ by IH

The last formula is S.

Case 6: $F = G \longleftrightarrow H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$F = G \longleftrightarrow H \equiv G_1 \longleftrightarrow H_1$ by IH

$\equiv ((G_1 \longrightarrow H_1) \longrightarrow ((H_1 \longrightarrow G_1) \longrightarrow \Box)) \longrightarrow \Box$ part 4 of the lemma

The last formula is S.

Case 7: $F = \forall x G$. By IH there is an S-formula G_1 such that $G \equiv G_1$.

$F = \forall G \equiv \forall G_1$ by IH

$\equiv \exists x (G_1 \longrightarrow \Box) \longrightarrow \Box$ part 5 of the lemma

The last formula is an S-formula.

Case 8: $F = \exists x G$. By IH there is an S-formula G_1 such that $G \equiv G_1$.

$F = \exists x G \equiv \exists x G_1$ by IH

The last formula is an S-formula.

Q.E.D.

Grading Criteria: 1. Listing the cases: 3 points

2. Case 1: 1 point
3. Case 2 (including part 1 of the lemma): 4 points
4. Case 3 (including part 2 of the lemma): 3 points
5. Case 4 (including part 3 of the lemma): 3 points
6. Case 5: 1.5 points
7. Case 6 (including part 4 of the lemma): 3 points
8. Case 7 (including part 5 of the lemma): 5 points

9.Case 8: 1.5 points

Question 4. (10 points)

Rectify the formula $F = \forall x \{\forall y [(P(x,y) \vee \exists z Q(z,y)) \wedge \exists y \neg P(y,z)] \wedge \forall z [\exists x (\neg P(z,x) \vee Q(y,z)) \wedge \forall z (P(x,z) \vee Q(z,y))]\}$.

Answer: x is quantified twice, so we relabel one of them.

$F \equiv \forall x\{\forall y[(P(x,y) \vee \exists z Q(z,y)) \wedge \exists y \neg P(y,z)] \wedge \forall z[\exists x_1(\neg P(z,x_1) \vee Q(y,z)) \wedge \forall z(P(x,z) \vee Q(z,y))]\}$

Now, y is free and also quantified 2 twice, so we relabel the quantified variables.

$F \equiv \forall x\{\forall y_1[(P(x,y_1) \vee \exists z Q(z,y_1)) \wedge \exists y_2 \neg P(y_2,z)] \wedge \forall z[\exists x_1(\neg P(z,x_1) \vee Q(y,z)) \wedge \forall z(P(x,z) \vee Q(z,y))]\}$

z is free and is quantified 3 times, so we relabel the quantified subformulas.

$F \equiv \forall x\{\forall y_1[(P(x,y_1) \vee \exists z_1 Q(z_1,y_1)) \wedge \exists y_2 \neg P(y_2,z)] \wedge \forall z_2[\exists x_1(\neg P(z_2,x_1) \vee Q(y,z_2)) \wedge \forall z_3(P(x,z_3) \vee Q(z_3,y))]\}$