COT 3420
Spring 2006

FINAL EXAM ANSWERS

QUESTIONS

Question 1. (15 points)
Write $E(F, 2)$ for $F = \forall x \forall y(\neg P(x, a) \wedge (P(f(x), y) \vee P(x, f(y))))$.
Answer: $D(F, 2) = \{a, f(a), f^2(a)\}$. Then
$E(F, 2) = \{F^M[x/a, y/a], F^M[x/a, y/f(a)], F^M[x/a, y/f^2(a)], F^M[x/f(a), y/a],$
$F^M[x/f(a), y/f(a)], F^M[x/f(a), y/f^2(a)], F^M[x/f^2(a), y/a], F^M[x/f^2(a), y/f(a)],$
$F^M[x/f^2(a), y/f^2(a)]\}$, where $F^M = \neg P(x, a) \wedge (P(f(x), y) \vee P(x, f(y)))$.
Grading Criteria: 5/3 points for each right formula
-1 point for each wrong formula
4 points for the correct D(F,2).

Question 2. (10 points)
Rectify the formula $F = \forall x[\exists y(P(x, y) \vee Q(x, z)) \wedge (\exists z \neg Q(x, z) \vee \neg Q(x, y))] \wedge \forall y[\exists x(P(x, y) \vee \neg P(a, x)) \wedge \forall y(\neg P(y, x) \vee Q(z, y))]$.
Answer: $F \equiv \forall x_1[\exists y_1(P(x_1, y_1) \vee Q(x_1, z)) \wedge (\exists z_1 \neg Q(x_1, z_1) \vee \neg Q(x_1, y))] \wedge \forall y_2[\exists x_2(P(x_2, y_2) \vee \neg P(a, x_2)) \wedge \forall y_3(\neg P(y_3, x) \vee Q(z, y_3))]$.
Grading Criteria: -1 point for each variable occurrence labeled incorrectly.

Question 3. (5 points)
Close the formula $F = \forall y \forall u F^M[x, y, z, u, v, w]$ where F^M, the matrix of F, has free occurrences of x, y, z, u, v, w.
Answer: $F \equiv_s \exists x \exists z \exists v \exists w F$
Grading Criteria: 1.25 points for each correct quantifier
-0.75 point for each redundant quantifier
-2 points for inserting the new quantifiers after $\forall y$

Question 4. (10 points)
Skolemize the formula $F = \exists x_1 \exists x_2 \forall x_3 \exists x_4 \exists x_5 \forall x_6 \exists x_7 F^M$, where F^M, the matrix of F, contains the constant a and the function symbols f and g.
Answer: $F \equiv_s \forall x_3 \forall x_6 F^M[x_1/b, x_2/c, x_4/h(x_3), x_5/i(x_3), x_7/j(x_3, x_6)]$.
Grading Criteria: 2.5 points for each correct substitution.

2. -1 point for using a, f, g.

Question 5. (10 points)

Find a prenex form for

$F = \neg\{\forall x[\exists y(P(x,y) \vee Q(x,y)) \wedge \exists z(\neg P(z,x) \vee Q(x,z))] \wedge \exists u[\forall v(P(v,u) \vee \neg Q(u,a)) \wedge \forall w(\neg P(w,u) \vee Q(b,w))]\}$.

Answer: $\exists x \forall u \exists v \exists w \forall y \forall z F^M$

Grading Criteria: -1 points for each wrong quantifier

-2 points for violating each of the precedences $\exists x$ before $\forall y$, $\exists x$ before $\forall z$, $\forall u$ before $\exists v$, $\forall u$ before $\exists w$.

-1.5 points for a missing quantifier

Question 6. (10 points)

Find a mgu for $S = \{P(x_1, f(x_2), g(x_1, x_3)), P(h(x_2), x_4, g(h(x_5), f(x_5))), P(x_6, f(h(x_7)), x_8)\}$ or prove that the set is not unifiable.

Answer: $\sigma = [x_1/h(h(x_7)), x_2/h(x_7), x_3/f(h(x_7)), x_4/f(h(x_7)), x_5/h(x_7), x_6/h(h(x_7)),$
$x_8/g(h(h(x_7)), f(h(x_7)))]$

Grading Criteria: 10/7 points for each correct substitution.

-3 points for leaving the mgu as a composition

-5 points for writing **not unifiable**

Question 7. (20 points)

Prove by first order resolution that $S = \{\{P(x,y), P(f(y),z), Q(x,y)\}, \{\neg P(y,a), Q(y,x)\},$
$\{R(h(x),x), \neg Q(x,a)\}, \{\neg R(x,y), \neg R(h(z),y), \neg Q(z,u)\}\}$ is unsatisfiable.
For each resolution step specify the relabelings, the unification set and the mgu. Do the minimal number of unifications.

Answer: The tree is displayed in Figure 1.

Grading Criteria: 1. 6.5 points for each resolution step: the relabelings 1 point, the unifying set 1 point, the mgu 2 points, the resolvent 2.5 points.

2. -2 points for each extra step

3. if one or both parents of the resolution step are wrong you loose anywhere from 3 to 6.5 points.

Question 8. (20 points)

We define the set of connectives $S = \{\downarrow, \forall x\}$ where $F \downarrow G = \neg(F \vee G)$. Prove, by structural induction on F, that S is adequate for FOL.

Proof:

Lemma 1. $\neg F \equiv F \downarrow F$

2.$F \vee G \equiv (F \downarrow G) \downarrow (F \downarrow G)$

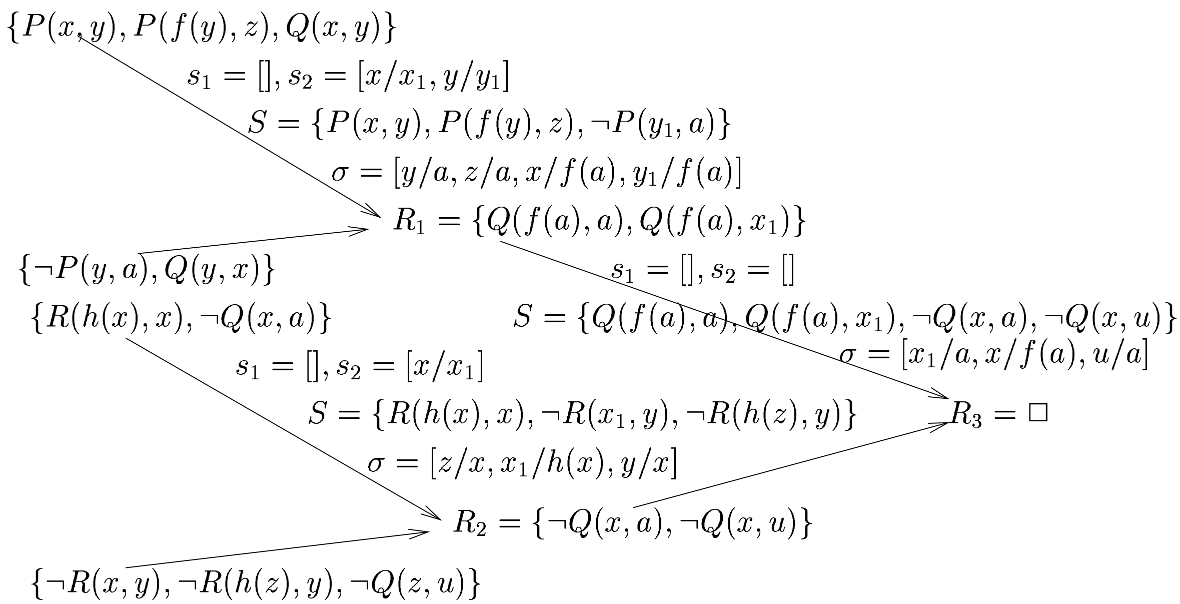

Figure 1: The resolution tree for Question 7

3. $F \wedge G \equiv (F \downarrow F) \downarrow (G \downarrow G)$
4. $F \longrightarrow G \equiv ((F \downarrow F) \downarrow G) \downarrow ((F \downarrow F) \downarrow G)$
5.$F \longleftrightarrow G \equiv ((F \downarrow F) \downarrow G) \downarrow ((G \downarrow G) \downarrow F)$
6. $\exists x F \equiv \forall x(F \downarrow F) \downarrow \forall x(F \downarrow F)$

Proof of the lemma:

1. $\neg F \equiv \neg F \wedge \neg F$ by idempotency
$\equiv \neg(F \vee F)$ DeMorgan's law
$= F \downarrow F$ definition of $\downarrow$
2. $F \vee G \equiv \neg\neg(F \vee G)$ double neg intro
$= \neg(F \downarrow G)$ definition of $\downarrow$
$\equiv (F \downarrow G) \downarrow (F \downarrow G)$ part 1
3. $F \wedge G \equiv \neg\neg F \wedge \neg\neg G$ double neg intro
$\equiv \neg(\neg F \vee \neg G)$ DeMorgan's law
$= \neg F \downarrow \neg G$ definition of $\downarrow$
$\equiv (F \downarrow F) \downarrow (G \downarrow G)$ part 1
4. $F \longrightarrow G \equiv \neg F \vee G$ $\longrightarrow$-elim
$\equiv (\neg F \downarrow G) \downarrow (\neg F \downarrow G)$ part 2
$\equiv ((F \downarrow F) \downarrow G) \downarrow ((F \downarrow F) \downarrow G)$ part 1
5. $F \longleftrightarrow G \equiv (F \longrightarrow G) \wedge (G \longleftrightarrow F)$ $\longleftrightarrow$-elim

$\equiv (\neg F \vee G) \wedge (\neg G \vee F)$ $\longrightarrow$-elim

$\equiv (\neg F \vee \neg\neg G) \wedge (\neg G \vee \neg\neg F)$ double neg intro

$\equiv \neg(F \wedge \neg G) \wedge \neg(G \wedge \neg F)$ DeMorgan's law twice

$\equiv \neg((F \wedge \neg G) \vee (G \wedge \neg F))$ DeMorgan's law

$= (F \wedge \neg G) \downarrow (G \wedge \neg F)$ definition of $\downarrow$

$\equiv (\neg\neg F \wedge \neg G) \downarrow (\neg\neg G \wedge \neg F)$ double neg intro

$\equiv \neg(\neg F \vee G) \downarrow \neg(\neg G \vee F)$ DeMorgan's law twice

$= (\neg F \downarrow G) \downarrow (\neg G \downarrow F)$ definition of $\downarrow$

$\equiv ((F \downarrow F) \downarrow G) \downarrow ((G \downarrow G) \downarrow F)$ part 1

6. $\exists x F \equiv \neg\neg x \exists x F$ double neg intro

$\equiv \neg \forall x \neg F$ pushing $\neg$ past $\exists x$

$\equiv \neg \forall x (F \downarrow F)$ part 1

$\equiv \forall x (F \downarrow F) \downarrow \forall x (F \downarrow F)$ part 1

Q.E.D. lemma

Proof: Let F be a formula.

Case 1: F is an atomic formula. Then F is an S-formula.

Case 2: $F = \neg G$. By IH there is an S-formula G_1 such that $G_1 \equiv G$. Then,

$F = \neg G$

$\equiv \neg G_1$ by IH

$\equiv G_1 \downarrow G_1$ Lemma, part 1

The last formula is an S-formula.

Case 3: $F = G \vee H$. By IH there are S-formulas G_1, H_1 such that $G_1 \equiv G$ and $H_1 \equiv H$. Then,

$F = G \vee H$

$\equiv G_1 \vee H_1$ by IH

$\equiv (G_1 \downarrow H_1) \downarrow (G_1 \downarrow H_1)$ Lemma, part 2

The last formula is an S-formula.

The rest of the cases are done the same way. **Q.E.D.**

Grading Criteria: Listing the 8 cases: 3 points.

Case 1: 1 point.

Cases 2,3,4,5,6,8: 2.5 points each.

Case 7: 1 point.

The cases include the corresponding part of the lemma.

Question 9. (15 points)

Display a formula F that is satisfied if and only if the structure has 4 elements.

Answer: Our formula F contain 4 distinct constants, a, b, c, d and every element in the universe must be equal to one of them. The formula below insures us that the constants are different.

$I = \neg E(a,b) \wedge \neg E(a,c) \wedge \neg E(a,d) \wedge \neg E(b,c) \wedge \neg E(b,d) \wedge \neg E(c,d)$

The formula J insures us that every element in the universe is equal to one of the 4 constants.

$J = \forall x(E(x,a) \vee E(x,b) \vee E(x,c) \vee E(x,d))$.

So $F = I \wedge J$.

The formula can also be written without constants as

$F = \exists x \exists y \exists z \exists u \forall v[\neg E(x,y) \wedge \neg E(x,z) \wedge \neg E(x,u) \wedge \neg E(y,z) \wedge \neg E(y,u) \wedge \neg E(z,u) \wedge (E(v,x) \vee E(v,y) \vee E(v,z) \vee E(v,u))$.

Grading Criteria:1. Using equality 5 points.

2. Using 4 constants: 5 points.

3. Just trying: 2 points.

Question 10. (20 points)

Let u and v two variables that do not occur in F, neither free nor as arguments to quantifiers. Prove that $\forall x \forall y F \models \forall u \forall v F[x/f(u,v), y/g(u)]$. Write your answer on a blank sheet of paper.

Hint: Use the Translation Lemma.

Proof: Let $\mathcal{A}$ be a model of $\forall x \forall y F$ and let U be the universe of $\mathcal{A}$. Then (1) is valid for all $d, e \in U$.

(1) $\mathcal{A}_{[x \leftarrow d][y \leftarrow e]}[F] = 1$

Now let us compute $\mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[F[x/f(u,v), y/g(u)]]$, where $a, b \in U$.

$\mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[F[x/f(u,v), y/g(u)]] = \mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[F[y/g(u)][x/f(u,v)]]$

$= \mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow \mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[f(u,v)]]}[F[y/g(u)]]$ by the Translation Lemma

$= \mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]]}[F[y/g(u)]]$ by evaluation $\mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[f(u,v)]$

$= \mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]][y \leftarrow \mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]]}[g(u)]]}[F]$ by the Translation lemma again

$= \mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]][y \leftarrow g^{\mathcal{A}}[a]]}[F]$ by evaluating $\mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]]}[g(u)][F]$

$= \mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[a,b]][y \leftarrow g^{\mathcal{A}}[a]]}[F]$ $\mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[a,b]][y \leftarrow g^{\mathcal{A}}[a]]}$ and $\mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[a,b]][y \leftarrow g^{\mathcal{A}}[a]]}$ agree on F since u and v are not free in F

$= 1$ by setting $d = f^{\mathcal{A}}[a,b]$ and $e = g^{\mathcal{A}}[a]$ in (1)

So,

(2) $\mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[F[x/f(u,v), y/g(u)]] = 1$

From the interpretation of $\forall v$ and $\forall u$, we get

(3) $\mathcal{A}[\forall u \forall v[F[x/f(u,v), y/g(u)]] = 1$. **Q.E.D.**

Grading Criteria: 1. Establishing (1) : 4 points
2. 2 points for each line of the proof of (2).