COT 3420
Section U1
FALL 2006

EXAM # 1 ANSWERS

Question 1. (28 points)
1. d 2. d 3. c 4. a 5. c 6. b 7. a 8. c 9. c 10. a 11. c 12. a 13. d 14. c

Grading criteria: 2 points for each correct answer.

Question 2. (25 points)

Do parts a and b.

a. (12 points) Prove that the set of connectives $S_1 = \{(F \vee \neg G) \wedge H\}$ is not adequate.

Proof: The idea is find a property that is true for all S_1 formulas, but false for some formulas in $FORM$. We claim that the truth assignment $\mathcal{A}$ that assigns 1 to all atoms is a model of all S_1 formulas. We prove it by structural induction on S_1.

Basis: If F is an atom P_i, then $\mathcal{A}[P_i] = 1$ from the definition of $\mathcal{A}$.

Inductive step: Assume that $F = (G \vee \neg H) \wedge I$.

We get (1) and (2) by IH on G and H.

(1) $\models_{\mathcal{A}} G$

(2) $\models_{\mathcal{A}} I$

Since $G \vee \neg H$ is a consequence of G, (1) implies (3).

(3) $\models_{\mathcal{A}} (G \vee \neg H)$

The formula $(G \vee \neg H) \wedge I$ is a consequence of the set $\{(G \vee \neg H), I\}$. So, (3) and (2) yield (4).

(3) $\models_{\mathcal{A}} ((G \vee \neg H) \wedge I)$.

Grading Criteria:

1. 3 points for guessing an unsatisfiable formula.
2. 2 points for the basis.
3. 7 points for the inductive step.

b. (13 points) Show that the set of connectives $S_2 = \{(F \vee \neg G) \wedge \neg H\}$ is adequate. You may use the fact that the sets $\{\neg F, F \wedge G\}$ and $\{\neg F, F \vee G\}$ are adequate.

We will show that the subset of S_2, $\{(F \lor \neg G) \land \neg F\}$ is adequate.

Proof:

Let $\phi[F, G] = (F \lor \neg G) \land \neg F$.

Then,

$\phi[F, F] = (F \lor \neg F) \land \neg F$ def of ϕ

$\equiv \mathbf{T} \land \neg F$ tautology law

$\equiv \neg F$ tautology law

So,

(1) $\sigma_\neg = \phi[F, F]$

Now let compute $\phi[\phi[G, F], \phi[G, F]]$.

$\phi[\phi[G, F], \phi[G, F]] \equiv \neg\phi[G, F]$ by (1)

$\equiv \neg((G \lor \neg F) \land \neg G)$ def of ϕ

$\equiv \neg(G \lor \neg F) \lor \neg\neg G$ De Morgan's law

$\equiv (\neg G \land \neg\neg F) \lor G$ De Morgans', double neg elim

$\equiv (\neg G \land F) \lor G$ double neg elim

$\equiv (\neg G \lor G) \land (F \lor G)$ distribute $\land$ over $\lor$

$\equiv \mathbf{T} \land (F \lor G)$ tautology law

$\equiv F \lor G$ tautology law

So,

(2) $\sigma_\lor = \phi[\phi[G, F], \phi[G, F]]$.

Grading Criteria:

1. 7 points for $\sigma_{/}= \phi[F, F]$
2. 6 points for $\sigma_\lor = \phi[\phi[G, F], \phi[G, F]]$.
3. If you did not find $\sigma_{/}$or $\sigma_\lor$ expressed as ϕ-formulas, you cannot get more than 4 points.

Question 3. (15 points)

Prove or disprove: If $F \longrightarrow G$ is a tautology, and F and G have no atoms in common, then at least one of $\neg F$ and G is a tautology.

First you must write Proof or Disproof and then provide the proof or the counter-example.

Proof: We assume that G is not a tautology and we will show that $\neg F$ is a tautology.

Since G is not a tautology, it has a countermodel $\mathcal{A}$.

(1) $\mathcal{A}[G] = 0$

Now let $\mathcal{B}$ be a truth assignment. We define the truth assignment $\mathcal{C}$ as follows:

$\mathcal{C}[P_i] = \mathcal{B}[P_i]$ if P_i is in F

$\mathcal{C}[P_i] = \mathcal{A}[P_i] \qquad$ if P_i is not in F

$\mathcal{C}$ agrees with $\mathcal{B}$ on F, so (2) holds.
(2) $\mathcal{C}[F] = \mathcal{B}[F]$
Since F and G have no atoms in common, $\mathcal{C}$ agrees with $\mathcal{A}$ on G.
(3) $\mathcal{C}[G] = \mathcal{A}[G]$
Since $F \longrightarrow G$ is a tautology,
(4) $\mathcal{C}[F \longrightarrow G] = 1$.
Now,
$\mathcal{C}[F \longrightarrow G] = \mathcal{C}[\neg F \vee G] \qquad \longrightarrow$ elimination
$= \mathcal{C}[\neg F]\boxed{\vee}\mathcal{C}[G] \qquad$ interpretation of $\vee$
$= \mathcal{B}[\neg F]\boxed{\vee}\mathcal{C}[G] \qquad$ by (2)
$= \mathcal{B}[F]\boxed{\vee}\mathcal{A}[G] \qquad$ by (3)
$= \mathcal{B}[F]\boxed{\vee}0 \qquad$ by (1)
$= \mathcal{B}[\neg F] \qquad$ from the tables of $\vee$
We collect the ends of this sequence of equalities and get (5).
(5) $\mathcal{C}[F \longrightarrow G] = \mathcal{B}[\neg F]$
From (4) and (5) we obtain (6).
(6) $\mathcal{B}[\neg F] = 1$
Since $\mathcal{B}$ is arbitrary, $\models \neg F$.

Grading Criteria:

1. If you wrote **Disproof** you get 2 points.
2. If you wrote **Proof** you get 5 points plus the points for the proof (up to 10).
3. The definition of $\mathcal{C}$ is worth 6 points, the rest of the proof 4 points.

Question 4. (14 points)

Construct a derivation tree of $\Box$ from $S = \{\{A, B, C\}, \{A, B, \neg C\}, \{A, \neg B, D\}, \{A, \neg B, \neg D\}, \{\neg A, D\}, \{\neg A, \neg D, E\}, \{\neg A, \neg D, \neg E\}\}$.

Answer: The tree is shown in Figure 1.
Grading Criteria: 1. 2 points for each correct resolution (up to 7) that leads to $\Box$.

2. -3 points for each incorret resolution step.

3. -4 points for presenting a derivation sequence instead of a derivation tree.

Question 5. (19 points)

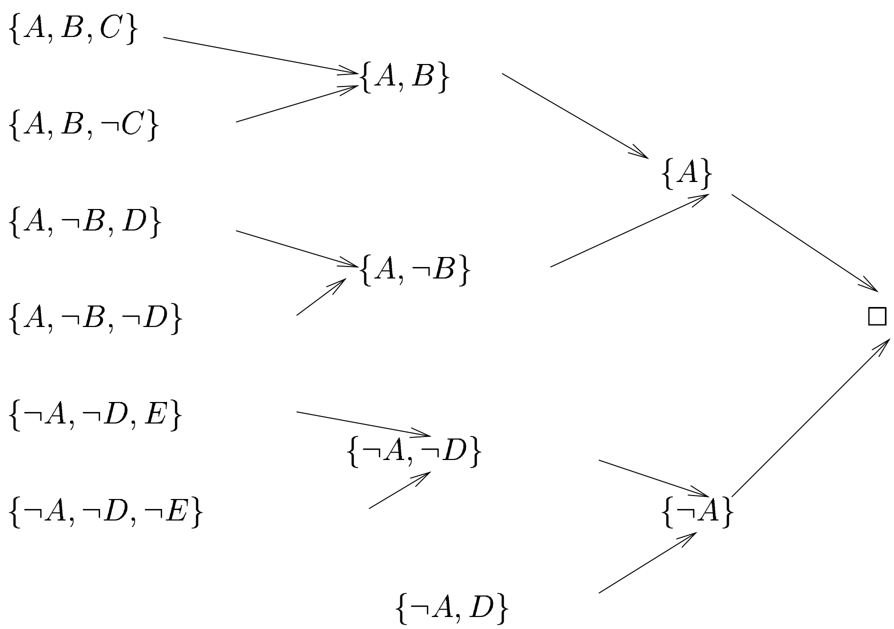

Figure 1: The answer to Question 4

Apply the algorithm given in the book to find a CNF for
$F = \{\neg[A \vee \neg(B \wedge C)] \longrightarrow [B \wedge \neg(C \wedge \neg D)]\}$.

Solution

$F = F_1 = \{\neg[A \vee \neg(B \wedge C)] \longrightarrow [B \wedge \neg(C \wedge \neg D)]\}$
$\equiv \neg\neg[A \vee \neg(B \wedge C)] \vee [B \wedge \neg(C \wedge \neg D)] = F_2$ $\quad\longrightarrow$-elim
$\equiv [A \vee \neg(B \wedge C)] \vee [B \wedge \neg(C \wedge \neg D)]$ double neg elim
$\equiv [A \vee \neg B \vee \neg C] \vee [B \wedge (\neg C \vee \neg\neg D)]$ DeMorgan's law, twice
$\equiv [A \vee \neg B \vee \neg C] \vee [B \wedge (\neg C \vee D)] = F_3$ double neg elim
$\equiv (A \vee \neg B \vee \neg C \vee B) \wedge (A \vee \neg B \vee \neg C \vee \neg C \vee D) = F_4$ distributivity
$\equiv (A \vee \neg B \vee \neg C \vee D) = F_5$ tautology law, idempotency

Grading Criteria: 1. You get credit up to the first line that contains an error.

1 correct line: 2 point
2 correct lines: 4 points
3 correct lines: 6 points
4 correct lines: 9 points
5 correct lines: 12 points
6 correct lines: 15 points

7 correct lines: 19 points

2. If you did not list the reasons, you loose 3 points.