COT 3420
Section U2
FALL 2006

EXAM # 1 ANSWERS

Question 1.(28 points)

1. a 2. b 3. c 4. a 5. b 6. d 7. c 8. a 9. c 10. a 11. b 12. a 13. b 14. a

Grading Criteria: 2 points for each correct answer.

Question 2. (15 points)

Let F, G, H be 3 formulas. We define the string $H[F/G]$ by the rules below.

1. If the string F does not occur in H, then $H[F/G] = H$.

2. If the string F occurs in H, $H[F/G]$ is the string obtained by replacing an occurrence of F by G.

Prove, by structural induction on H, that $H[F/G]$ is a formula.

Proof:

Case 1: H is an atomic formula.

Since H is a symbol, the only subformula of F is F itself. If $F = H$, then $H[F/G] = G$, a formula. If $F \neq H$, then $H[F/G] = H$, a formula. So, in both cases, $H[F/G]$ is a formula.

Case 2: $H = \neg I$.

The subformulas of H are H and the subformulas of I. We have 3 subcases:

Subcase 2.1: F does not occur in H.

Then, $H[F/G] = H$, a formula.

Subcase 2.2: F occurs in H, and $H = F$.

Then, $H[F/G] = G$, a formula.

Subcase 2.3: F occurs in H and $F \neq H$.

Then, all occurrences of F are in I. So, $H[F/G] = \{\neg I\}[F/G] = \neg I[F/G]$, i.e. replacing F by G in H is the same as replacing F by G in I, and then puting $\neg$ in front of $I[F/G]$. By IH, $I[F/G]$ is a formula, so $\neg I[F/G]$ is also a formula.

Cases 3,4,5,6: $H = (ICJ)$, where C is a binary connective.

The subformulas of H are H and the subformulas of I and J. We have 4 subcases.

Subcase 3.1: F does not occur in H.

Then, $H[F/G] = H$, a formula.

Subcase 2.2: F occurs in H, and $H = F$.

Then, $H[F/G] = G$, a formula.

Subcase 2.3: The F that is being replaced occurs in I.

Then, $H[F/G] = (ICJ)[F/G] = (I[F/G]CJ)$, i.e. replacing F by G in H is the same as replacing F by G in H, and then forming the string $(I[F/G]CJ)$. By IH, $I[F/G]$ is a formula, so $(I[F/G]CJ)$ is also a formula.

Subcase 2.4: The F that is being replaced occurs in J.

The proof is similar to Subcase 2.3.

Q.E.D.

Grading Criteria: 1. Listing the cases : 2 points

2. Case 1: 2 points
3. Case 2: 5 points
4. Case 3: 6 points

Question 3. (25 points)

Proof or disproof: If the formulas $F \wedge G \wedge H \wedge I \wedge J$, $G \wedge H \wedge I \wedge J \wedge K$, $H \wedge I \wedge J \wedge K \wedge F$, $I \wedge J \wedge K \wedge F \wedge G$, $J \wedge K \wedge F \wedge G \wedge H$, $K \wedge F \wedge G \wedge H \wedge I$ are satisfiable, then $F \wedge G \wedge H \wedge I \wedge J \wedge K$ is also satisfiable.

Disproof: Let $F = P_1 \vee P_2 \vee P_3$, $G = P_1 \vee P_2 \vee \neg P_3$, $H = P_1 \vee \neg P_2 \vee P_3$, $I = P_1 \vee \neg P_2 \vee \neg P_3$, $J = \neg P_1 \vee P_2$, $K = \neg P_1 \vee \neg P_2$, and $L = F \wedge G \wedge H \wedge I \wedge J$, $M = G \wedge H \wedge I \wedge J \wedge K$, $N = H \wedge I \wedge J \wedge K \wedge F$, $O = I \wedge J \wedge K \wedge F \wedge G$, $P = J \wedge K \wedge F \wedge G \wedge H$, $Q = K \wedge F \wedge G \wedge H \wedge I$, $R = F \wedge G \wedge H \wedge I \wedge J \wedge K$.

The truth table for these formulas is shown below.

P_1	P_2	P_3	F	G	H	I	J	K	L	M	N	O	P	Q	R
0	0	0	0	1	1	1	1	1	0	1	0	0	0	0	0
0	0	1	1	0	1	1	1	1	0	0	1	0	0	0	0
0	1	0	1	1	0	1	1	1	0	0	0	1	0	0	0
0	1	1	1	1	1	0	1	1	0	0	0	0	1	0	0
1	0	0	1	1	1	1	0	1	0	0	0	0	0	1	0
1	0	1	1	1	1	1	0	1	0	0	0	0	0	1	0
1	1	0	1	1	1	1	1	0	1	0	0	0	0	0	0
1	1	1	1	1	1	1	1	0	1	0	0	0	0	0	0

As it can be seen from the above table, L, M, N, O, P, Q are satisfiable, but R is not.

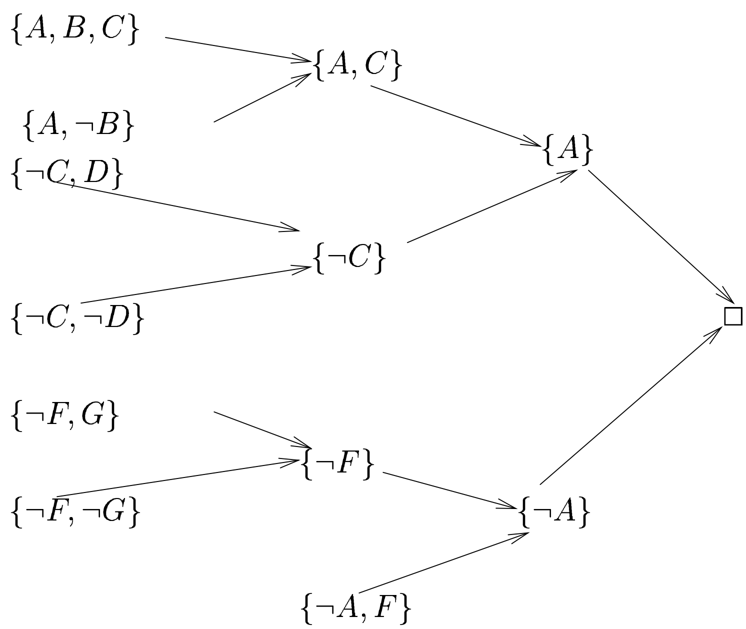

Figure 1: The answer to Question 4

Grading Criteria:

1. If you wrote **Proof** you get 3 points.
2. If you wrote **Disproof** or **counter-example** you get 6 points plus the points for selecting F, G, H, I, J, K and the table.
3. The selection of F, G, H, I, J, K is worth 2 points each.
4. If you chose F, G, H, I, J, K you may get up to 7 points for the table.

Question 4. (14 points)

Construct a derivation tree of $\square$ from
$S = \{\{A, B, C\}, \{A, \neg B\}, \{\neg C, D\}, \{\neg C, \neg D\},$
$\{\neg F, \neg G\}, \{\neg F, G\}, \{\neg A, F\}\}$.

Answer: The tree is shown in Figure 1.

Grading Criteria: 1. 7/30 points for each correct resolution (up to 6) that leads to $\square$.

2. -3 points for each incorrect resolution step.

3. -4 points for presenting a derivation sequence instead of a derivation tree.

Question 5. (19 points)

Apply the algorithm given in the book to find a CNF for $F = \neg[((A \vee B) \wedge C) \longleftrightarrow \neg(B \vee D)]$.

Solution:

$F = \neg[((A \vee B) \wedge C) \longleftrightarrow \neg(B \vee D)]$

$\equiv \neg\{[((A \vee B) \wedge C) \longrightarrow \neg(B \vee D)] \wedge [\neg(B \vee D) \longrightarrow ((A \vee B) \wedge C)]\} = F_1$ $\longleftrightarrow$-elim

$\equiv \neg\{[\neg((A \vee B) \wedge C) \vee \neg(B \vee D)] \wedge [\neg\neg(B \vee D) \vee ((A \vee B) \wedge C)]\} = F_2$ $\longrightarrow$-elim

$\equiv \neg[\neg((A \vee B) \wedge C) \vee \neg(B \vee D)] \vee \neg[\neg\neg(B \vee D) \vee ((A \vee B) \wedge C)]$ DeMorgan's law

$\equiv [\neg\neg((A \vee B) \wedge C) \wedge \neg\neg(B \vee D)] \vee [\neg\neg\neg(B \vee D) \wedge \neg((A \vee B) \wedge C)]$ DeMorgan's law twice

$\equiv [((A \vee B) \wedge C) \wedge (B \vee D)] \vee [\neg(B \vee D) \wedge (\neg(A \vee B) \vee \neg C)]$ double neg elim 3 times, De Morgan's law

$\equiv [(A \vee B) \wedge C \wedge (B \vee D)] \vee [\neg B \wedge \neg D \wedge ((\neg A \wedge \neg B) \vee \neg C)] = F_3$ De Morgan's law twice

$\equiv [(A \vee B) \wedge C \wedge (B \vee D)] \vee [\neg B \wedge \neg D \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg C)]$ distributivity

$\equiv (A \vee B \vee \neg B) \wedge (A \vee B \vee \neg D) \wedge (A \vee B \vee \neg A \vee \neg C) \wedge (A \vee B \vee \neg B \vee \neg C) \wedge (C \vee \neg B) \wedge (C \vee \neg D) \wedge (C \vee \neg A \vee \neg C) \wedge (C \vee \neg B \vee \neg C) \wedge (B \vee D \vee \neg B) \wedge (B \vee D \vee \neg D) \wedge (B \vee D \vee \neg A \vee \neg C) \wedge (B \vee D \vee \neg B \vee \neg C) = F_4$ generalized distributivity

$\equiv (A \vee B \vee \neg D) \wedge (\neg B \vee C) \wedge (C \vee \neg D) \wedge (\neg A \vee B \vee \neg C \vee D) = F_5$ tautology elim

Note At line 8, we can simplify $\neg B \wedge \neg D \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg C)$ to $\neg B \wedge \neg D \wedge (\neg A \vee \neg C)$ by absorbtion

Grading Criteria: 1. You get credit up to the first line where a mistake was made, or the end of your derivation, whatever comes first.

for 1 correct line you get 1 point
for 2 correct lines you get 2 points
for 3 correct lines you get 3 points
for 4 correct lines you get 4 points
for 5 correct lines you get 6 points
for 6 correct lines you get 9 points
for 7 correct lines you get 12 points
for 8 correct lines you get 14 points

for 9 correct lines you get 17 points

for 10 correct lines you get 19 points

2. You loose up to 3 points for not writing the reasons.