COT 3420
FALL 2006
Section U1

EXAM # 2 ANSWERS

QUESTIONS

Question 1. (27 points)

The universe of structure $\mathcal{A}$ is $\{3, 4, 5, 6\}$. The $\mathcal{A}$ interpretations of a, x, and y are $a^{\mathcal{A}} = 3$, $x^{\mathcal{A}} = 5$, $y^{\mathcal{A}} = 6$. The tables for the functions $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ and the predicate $P^{\mathcal{A}}$ are shown in Figure 1. Evaluate the terms and formulas below. Do not show your work, just write the answer after the equal sign.

1. $\mathcal{A}[f(x)] = 3$
2. $\mathcal{A}[g(x, y)] = 5$
3. $\mathcal{A}[g(g(a, y), f(x))] = 3$
4. $\mathcal{A}[P(x, f(a))] = 0$
5. $\mathcal{A}[E(f(y), g(x, a))] = 0$
6. $\mathcal{A}[\forall x P(x, f(x))] = 0$
7. $\mathcal{A}[\forall y P(f(y), y)] = 1$
8. $\mathcal{A}[\forall x \exists y P(y, x)] = 1$
9. $\mathcal{A}[\exists x \forall y \neg P(x, y)] = 1$

Grading Criteria: 3 points for each correct answer.

Question 2. (20 points)

u	$f^{\mathcal{A}}[u]$
3	3
4	5
5	3
6	5

$f^{\mathcal{A}}[u]$

$u \backslash v$	3	4	5	6
3	3	3	3	3
4	3	4	5	6
5	3	5	3	5
6	3	6	5	4

$g^{\mathcal{A}}[u, v]$

$u \backslash v$	3	4	5	6
3	1	0	1	0
4	0	0	0	0
5	0	1	0	1
6	0	0	0	0

$P^{\mathcal{A}}[u, v]$

Figure 1: Tables for Question 1

Show that the set of connectives $S = \{\exists x \neg F \wedge G\}$ is adequate for FOL. Hint: Let $\sigma[x, F, G] = \exists x \neg F \wedge G$. Find 3 formulas $\phi_{\neg}[F]$, $\phi_{\wedge}[F, G]$ and $\phi_{\exists x}[F]$ that contain only the σ-operator, the meta-variable(s) specified in the brackets, and atomic formulas, such that

$\phi_{\neg}[F] \equiv \neg F$, $\phi_{\wedge}[F, G] \equiv (F \wedge G)$ and $\phi_{\exists x}[F] \equiv \exists x F$,

and use the fact that $T = \{\neg F, F \wedge G, \exists x F\}$ is adequate.

Solution:1. We take $\phi_{\neg}[F] = \sigma[x, F, E(z, z)]$, where x is a variable that is not free in F.

$\sigma[x, F, E(z, z)] = \exists x \neg F \wedge E(z, z)$ definition of $\sigma[x, F, E(z, z)]$

$\equiv \exists x \neg F \wedge \mathbf{T}$ $E(z, z)$ is a tautology

$\equiv \exists x \neg F$ tautology law

$\equiv \neg F$ $\exists x$ is redundant

2. We take $\phi_{\wedge}[F, G] = \sigma[x, \phi_{\neg}[F], G]$, where $\phi_{\neg}[F]$ is the formula defined above and x is a variable that is not free in F.

$\sigma[x, \phi_{\neg}[F], G] = \exists x \neg \phi_{\neg}[F] \wedge G$ definition of $\sigma[x, \phi_{\neg}[F], G]$

$\equiv \exists x \neg\neg F \wedge G$ Part 1

$\equiv \exists x F \wedge G$ double negation elim

$\equiv F \wedge G$ eliminate redundant quantifiers

3. We take $\phi_{\exists x}[F] = \sigma[x, \phi_{\neg}[F], E(z, z)]$.

$\sigma[x, \phi_{\neg}[F], E(z, z)] = \exists x \neg \phi_{\neg}[F] \wedge E(z, z)$ definition of $\sigma[x, \phi_{\neg}[F], E(z, z)]$

$\equiv \exists x \neg\neg F \wedge E(z, z)$ construction of $\phi_{\neg}[F]$

$\equiv \exists x F \wedge E(z, z)$ double negation elim

$\equiv \exists x F \wedge \mathbf{T}$ $E(z, z)$ is a tautology

$\equiv \exists x F$ tautology law

Grading Criteria: 1. Finding $\phi_{\neg}[F]$: 8 points; the proof is worth 4 points

2. Finding $\phi_{\wedge}[F, G]$: 6 points; the proof is worth 3 points
3. Finding $\phi_{\exists x}[F]$: 6 points; the proof is worth 3 points

Question 3 (10 points)

Rectify the formula

$F = [\forall x \neg P(x,x) \wedge \forall x \forall y \exists y P(x,y)] \wedge [\forall x \forall y \forall z((\neg P(x,y) \vee \neg P(y,z)) \vee P(x,z)) \wedge P(y,y)]$

Write your answer below.

$F \equiv [\forall x \neg P(x,x) \wedge \forall x_1 \exists y_1 P(x_1,y_1)] \wedge [\forall x_2 \forall y_2 \forall z((\neg P(x_2,y_2) \vee \neg P(y_2,z)) \vee P(x_2,z)) \wedge P(y,y)]$

Grading Criteria: 1. -1 point for each wrong symbol.

2. -2 points for eliminating the wrong quantifier.

Question 4 (8 points)

Skolemize the formula $F = \exists x \forall y \exists z \forall u \forall v \exists w F^M$, where F^M is the matrix of F. The matrix of F contains the constants a and b and the function symbols f and g. Don't show your work, just write the Skolemized formula.

Write the substitutions as [variable/term]. Display the answer below.

$F \equiv_s \forall y \forall u \forall v F^M[x/c, z/h(y), w/i(y,u,v)]$

Grading Criteria: 1. Using a, b, f, g as replacements for x, z, w: -1 point per occurrence

2. Using the wrong terms (like the arity, or the arguments) for replacement : at least -1.5 points per occurrence.

3. Replacing the wrong variable: at least -2 points per variable.

4. Otherwise: x/c is worth 2 points, $z/h(y)$ 2.5 points, and $w/i(y,u,v)$ 3.5 points

Question 5 (10 points)

Find a prenex form for

$F = [\forall x(\exists y P(x,y) \vee \forall z Q(x,z)) \wedge \exists u(\forall v \neg P(u,v) \vee \exists w \neg Q(w,u))]$.

Don't show your work, just write the prenex form. Display your answer below.

$F \equiv \exists u \exists w \forall x \exists y \forall z \forall v[(P(x,y) \vee Q(x,z)) \wedge (\neg P(u,v) \vee \neg Q(w,u))]$

In fact any quantifier permutation where $\forall x$ is before $\exists y$ and $\exists u$ is before $\forall v$ is good.

Grading Criteria: -1 point for changing the quantifier symbol.

Question 6 (5 points)

Close the formula $F = \forall z \exists v P(x, y, z, u, v)$.

Write your answer below.

Answer: $F \equiv_s \exists x \exists y \exists u F$.

Grading Criteria: 5/3 points for each existential quantifier.

Question 7 (20 points)

Let F be a rectified formula and u, v be two variables that do not occur in F. Prove that $\forall x \forall y F \models \forall u \forall v F[x/f(u), y/g(v)]$. Here, f and g are unary functions, and x, y, u, v are 4 different variables. Write your answer on a blank sheet of paper.

Hint: Apply the translation lemma.

Solution: Let $\mathcal{A}$ be a model of $\forall x \forall y F$ and U be the universe of $\mathcal{A}$.

From the interpretation of $\forall x \forall y F$ we get that for all $d, e \in U$, (1) holds.

(1) $\mathcal{A}_{[x \leftarrow d][y \leftarrow e]}[F] = 1$

Now let us evaluate $\mathcal{A}[\forall u \forall v F[x/f(u), y/g(v)]]$.

$\mathcal{A}[\forall u \forall v F[x/f(u), y/g(v)]] = 1$

iff for all $p \in U$, $\mathcal{A}_{[u \leftarrow p]}[\forall v F[x/f(u), y/g(v)]] = 1$ interpretation of $\forall u$

iff for all $p \in U$, forall $q \in U$, $\mathcal{A}_{[u \leftarrow p][v \leftarrow q]}[F[x/f(u), y/g(v)]] = 1$ interpretation of $\forall v$

iff for all $p \in U$, for all $q \in U$, $\mathcal{A}_{[u \leftarrow p][v \leftarrow q]}[F[y/g(v)][x/f(u)]] = 1$ since $g(v)$ does not contain x, $[x/f(u), y/g(v)] = [y/g(v)][x/f(u)]$

iff for all $p \in U$, for all $q \in U$, $\mathcal{A}_{[u \leftarrow p][v \leftarrow q][x \leftarrow \mathcal{A}_{[u \leftarrow p][v \leftarrow q]}[f(u)]]}[F[y/g(v)]] = 1$ $f(u)$ is free for x in $F[y/g(v)]$, so we apply the translation lemma

iff for all $p \in U$, for all $q \in U$, $\mathcal{A}_{[u \leftarrow p][v \leftarrow q][x \leftarrow f^{\mathcal{A}}[p]]}[F[y/g(v)]] = 1$ evaluation of $\mathcal{A}_{[u \leftarrow p][v \leftarrow q]}[f(u)]$

iff for all $p \in U$, for all $q \in U$, $\mathcal{A}_{[u \leftarrow p][v \leftarrow q][x \leftarrow f^{\mathcal{A}}[p]][y \leftarrow \mathcal{A}_{[u \leftarrow p][v \leftarrow q][x \leftarrow f^{\mathcal{A}}[p]]}[g(v)]]}[F] = 1$ $g(v)$ is free for y in F

iff for all $p \in U$, for all $q \in U$, $\mathcal{A}_{[u \leftarrow p][v \leftarrow q][x \leftarrow f^{\mathcal{A}}[p]][y \leftarrow g^{\mathcal{A}}[q]]}[F] = 1$ evaluating $\mathcal{A}_{[u \leftarrow p][v \leftarrow q][x \leftarrow f^{\mathcal{A}}[p]]}[g(v)]$

iff $p \in U$, for all $q \in U$, $\mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[p]][y \leftarrow g^{\mathcal{A}}[q]]}[F] = 1$ u, v do not occur in F, so $\mathcal{A}_{[u \leftarrow p][v \leftarrow q][x \leftarrow f^{\mathcal{A}}[p]][y \leftarrow g^{\mathcal{A}}[q]]}$ and $\mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[p]][y \leftarrow g^{\mathcal{A}}[q]]}$ agree on F

Since $\mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[p]][y \leftarrow g^{\mathcal{A}}[q]]}[F] = 1$ by (1), the last statement is true.

Grading Criteria: 1. Establishing: (1) 5 points.

2. The rest of the derivation: 15 points.