COT 3420
FALL 2006
Section U2

EXAM # 2 ANSWERS

Question 1. (27 points)

The universe of structure $\mathcal{A}$ is $\{3, 4, 5, 6\}$. The $\mathcal{A}$ interpretations of a, x, and y are $a^{\mathcal{A}} = 5$, $x^{\mathcal{A}} = 6$, $y^{\mathcal{A}} = 3$. The tables for the functions $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ and the predicate $P^{\mathcal{A}}$ are shown in Figure 1. Evaluate the terms and formulas below. Do not show your work, just write the answer after the equal sign.

1. $\mathcal{A}[f(x)] = 5$

2. $\mathcal{A}[g(x, y)] = 3$

3. $\mathcal{A}[g(g(a, y), f(x))] = 3$

4. $\mathcal{A}[P(x, f(a))] = 1$

5. $\mathcal{A}[E(f(y), g(x, a))] = 0$

6. $\mathcal{A}[\forall x P(f(x), x)] = 1$

7. $\mathcal{A}[\forall y P(y, f(y))] = 0$

8. $\mathcal{A}[\forall x \exists y P(y, x)] = 1$

9. $\mathcal{A}[\forall x \exists y P(x, y)] = 0$

Grading Criteria: 3 points for each correct answer.

Question 2. (15 points)

Show that the set of connectives $S = \{\forall x \neg F \vee \exists y \neg G\}$ is adequate for FOL.

u	$f^{\mathcal{A}}[u]$
3	4
4	4
5	6
6	5

$f^{\mathcal{A}}[u]$

$u \backslash v$	3	4	5	6
3	3	3	3	3
4	3	4	5	6
5	3	5	3	5
6	3	6	5	4

$g^{\mathcal{A}}[u,v]$

$u \backslash v$	3	4	5	6	
3	0	0	0	0	
4	1	1	0	0	
5	0	0	0	1	
6	0	0	1	1	

$P^{\mathcal{A}}[u,v]$

Figure 1: Tables for Question 1

Hint: Let $\sigma[x,F,y,G] = \forall x \neg F \vee \exists y \neg G$. Find 3 formulas $\phi_{\neg}[F]$, $\phi_{\vee}[F,G]$ and $\phi_{\forall x}[F]$ that contain only the σ-operator, the meta-variable(s) specified in the brackets, and atomic formulas, such that

$\phi_{\neg}[F] \equiv \neg F$, $\phi_{\vee}[F,G] \equiv (F \vee G)$ and $\phi_{\forall x}[F] \equiv \forall x F$,

and use the fact that $T = \{\neg F, F \vee G, \forall x F\}$ is adequate.

Solution:1. We take $\phi_{\neg}[F] = \sigma[x,F,y,E(z,z)]$, where x is a variable that is not free in F.

$\sigma[x,F,y,E(z,z)] = \forall x \neg F \vee \exists y \neg E(z,z)$ definition of $\sigma[x,F,y,E(z,z)]$

$\equiv \forall x \neg F \vee \exists y \neg \mathbf{T}$ $E(z,z)$ is a tautology

$\equiv \forall x \neg F \vee \exists y \Box$ tautology law

$\equiv \forall x \neg F \vee \Box$ remove the redundant quantifier $\exists y$

$\equiv \forall x \neg F$ contradiction law

$\equiv \neg F$ $\exists x$ is redundant

2. We take $\phi_{\vee}[F,G] = \sigma[x,\phi_{\neg}[F],y,\phi_{\neg}[G]]$, where $\phi_{\neg}[F]$ is the formula defined above, x is not free in F and y is not free in G.

$\sigma[x,\phi_{\neg}[F],y,\phi_{\neg}[G]] = \forall x \neg \phi_{\neg}[F] \vee \exists \neg \phi_{\neg}[G]$ definition of $\sigma[x,\phi_{\neg}[F],y,\phi_{\neg}[G]]$

$\equiv \forall x \neg\neg F \vee \exists y \neg\neg G$ by the construction of $\phi_{\neg}$

$\equiv \forall x F \vee \exists y G$ double negation elim twice

$\equiv F \vee G$ eliminate redundant quantifiers

3. We take $\phi_{\forall x}[F] = \sigma[x,\phi_{\neg}[F],y,E(z,z)]$.

$\sigma[x,\phi_{\neg}[F],y,E(z,z)] = \forall x \neg \phi_{\neg}[F] \vee \exists y \neg E(z,z)$ definition of $\sigma[x,\phi_{\neg}[F],y,E(z,z)]$

$\equiv \forall x \neg\neg F \vee \exists y \neg E(z,z)$ construction of $\phi_{\neg}[F]$

$\equiv \forall x \neg\neg F \vee \exists y \neg \mathbf{T}$ $E(z,z)$ is a tautology

$\equiv \forall x \neg\neg F \vee \exists y \Box$ tautology law

$\equiv \forall x \neg\neg F \vee \Box$ eliminate the redundant quantifier $\exists y$

$\equiv \forall x \neg\neg F$ contradiction law

$\equiv \forall x F$ double negation elimination

Grading Criteria: 1. Finding $\phi_{\neg}[F]$: 8 points; the proof is worth 4 points. If you did not say that x is not free in F you loose 1 point.

2. Finding $\phi_{\wedge}[F, G]$: 6 points; the proof is worth 3 points. If you did not say that x is not free in F and y is not free in G, you loose 1 point.

3. Finding $\phi_{\exists x}[F]$: 6 points; the proof is worth 3 points

Question 3 (10 points)

Rectify the formula

$F = \forall x[\forall y(P(x, y) \vee Q(x, z)) \wedge \exists z \exists x \forall y(\neg P(x, y) \vee \forall z \neg Q(x, z))]$

Answer: $F \equiv \forall x[\forall y(P(x, y) \vee Q(x, z)) \wedge \exists x_1 \forall y_1(\neg P(x_1, y_1) \vee \forall z_1 \neg Q(x_1, z_1))]$

Grading Criteria:1. -1 points for each wrong symbol.

2. -2 points for not eliminating, or eliminating the wrong redundant quantifier.

Question 4 (8 points)

Skolemize the formula $F = \exists x \exists y \forall z \exists u \forall v \exists w F^M$, where F^M is the matrix of F. The matrix of F contains the constants a and b and the function symbols f and g. Don't show your work, just write the Skolemized formula.

Write the substitutions as [variable/term]. Display the answer below.

Answer: $F \equiv_s \forall z \forall v F^M[x/c, y/d, u/h(z), w/i(z, v)]$

1. Using a, b, f, g as replacements for x, y, u, w: -1 point per occurrence

2. Using the wrong terms (like the arity, or the arguments) for replacement : at least -1.5 points per occurrence.

3. Replacing the wrong variable: at least -2 points per variable.

4. Overloading a function symbol: -1 point for each occurrence

5. Otherwise: $x/c, y/d$ are worth 1.75 points each, $u/h(z)$ 2 points, and $w/i(z, v)$ 2.5 points.

Question 5 (10 points)

Find a prenex form for

$F = \forall x[\exists y P(x, y) \vee \forall z Q(x, z)] \wedge \exists u[\forall v \neg P(u, v) \wedge \exists w \neg Q(w, u)]$.

Don't show your work, just write the prenex form. Display your answer below.

Answer: $F \equiv \exists u \exists w \forall x \exists y \forall z \forall v \{[P(x, y) \vee Q(x, z)] \wedge [\neg P(u, v) \wedge \neg Q(w, u)]$.

In fact any quantifier permutation where $\exists u$ is before $\forall v$ and $\forall x$ is before $\exists y$ is a prenex form.

Grading Criteria: 1. -1 point for changing the quantifier symbol.

2. -2 points for omitting a quantifier.

Question 6 (5 points)

Close the formula $F = \forall z \exists v P(x, y, z, u, v)$.

Write your answer below.

Answer: $F \equiv_s \exists x \exists y \exists u F$.

Grading Criteria: 1. 5/3 points for each existential quantifier.

2. Inserting the existential quantifiers after $\forall z \exists v$: -2 points.

Question 7 (20 points)

a. (10 points)

Let x be free for y in F. Then, prove that $\forall x \forall y F \models \forall x F[y/x]$.

Hint: Apply the translation lemma.

Solution:

Let $\mathcal{A}$ be a model of $\forall x \forall y F$ and let U be the universe of $\mathcal{A}$.

Then, from the interpretation of $\forall x \forall y F$, (1) holds for all $d, e \in U$.

(1) $\mathcal{A}_{[x \leftarrow d][y \leftarrow e]}[F] = 1$, from the interpretation of $\forall x \forall y F$.

Now let us compute $\mathcal{A}_{[x \leftarrow d]}[F[y/x]]$.

$\mathcal{A}_{[x \leftarrow d]}[F[y/x]] = \mathcal{A}_{[x \leftarrow d][y \leftarrow \mathcal{A}_{[x \leftarrow d]}[x]]}[F]$ x is free for y in F, so we apply the translation lemma

$= \mathcal{A}_{[x \leftarrow d][y \leftarrow d]}[F]$ evaluation of $\mathcal{A}_{[x \leftarrow d]}[x]$

$= 1$ by (1)

Since $\mathcal{A}_{[x \leftarrow d]}[F[y/x]] = 1$ for all $d \in U$, $\mathcal{A}[\forall x F[y/x]] = 1$ by the interpretation of $\forall x$.

Grading Criteria: you get 3 points for establishing (1).

b. (10 points)

Show that the consequence does not always hold when x is not free for y in F.

Let $F = \exists x \neg E(x, y)$. Of course, x is not free in F, so $\forall x \forall y F \equiv \forall y F = \forall y \exists x \neg E(x, y)$. This formula is satisfied by all structures whose domain has more than 1 element. Let $\mathcal{A}$ be one of these structures. Let us evaluate $\forall x F[y/x]$.

$\forall x F[y/x] = \forall x \exists x \neg E(x, x)$

$\equiv \exists x \neg E(x, x)$ $\forall x$ is redundant

$\equiv \exists x \neg \mathbf{T}$ $E(x, x)$ is a tautology

$\equiv \exists x \Box$ tautology law

$\equiv \Box$ eliminate the redundant quantifier $\exists x$

So, the LHS of $\forall x \forall y F \models \forall x F[y/x]$ has models, but the RHS does not. Contradiction.