COT 3420
Section U1
SPRING 2007

EXAM # 1 ANSWERS

QUESTIONS

Question 1. (28 points)

1\. d 2. c 3. a 4. c 5. c 6. a 7. b 8. b 9. d 10. b 11. b 12. a 13. a 14. c

Grading Criteria: 2 points for each correct answer.

Question 2. (25 points)

Do parts a and b.

a. (13 points) Prove that the set of connectives $S_1 = \{F \vee \neg G\}$ is not adequate.

b. (12 points) Show that the set of connectives $S_2 = \{(F \wedge \neg G), \{\mathbf{T}\}\}$ is adequate. You may use the fact that the sets $\{\neg F, F \wedge G\}$ and $\{\neg F, F \vee G\}$ are adequate.

Part a

We prove the claim below, by structural induction on S_1.

Claim: The truth assignment $\mathcal{A}$ defined below, satisfies every S_1 formula.

(1) for all $i \in N$, $\mathcal{A}[P_i] = 1$

Proof:

Case 1: F is an atom, say P_i.

Then, $\mathcal{A}[P_i] = 1$.

Case 2: $F = G \vee \neg H$. By IH, $\mathcal{A}[G] = 1$.

Then, $\mathcal{A}[F] = \mathcal{A}[G \vee \neg H]$ $F = G \vee \neg H$

$= \mathcal{A}[G] \boxed{\vee} \mathcal{A}[\neg H]$ interpretation of $\vee$

$= 1 \boxed{\vee} \mathcal{A}[\neg H]$ by IH

$= 1$ table of $\boxed{\vee}$

Q.E.D.

This tells us that formulas like $\neg P_i$ have no S_1 equivalent, so S_1 is not adequate.

Another Proof: This time let us prove, by contradiction, that there is no S_1 formula equivalent to $\Box$. Assume that there are S_1 formulas equivalent to it. Then, let F be such a formula of **minimal length**.

F cannot be an atom, since very atom is satisfiable.

Then $F = G \vee \neg H$ for some S_1 formulas G and H. Since $F \equiv \Box$, $G \equiv \Box$. But this contradicts the assumption that F has minimal length.

Grading Criteria: Guessing $\mathcal{A}$ or $\Box$: 4 points

The proof: 9 points.

Part b: We prove, by structural induction on $T = \{\neg F, F \wedge G\}$, that every T-formula has an equivalent S_2 formula. For clarity purposes, let us write $F \uparrow G$ for $(F \wedge \neg G)$.

Case 1: F is an atom.

Then, F is an S_2 formula.

Case 2: $F = \neg G$.

By IH there is an S_2 formula G_1 such that $G \equiv G_1$. Then,

$F = \neg G$

$\equiv \neg G_1$ by IH

$\equiv \mathbf{T} \wedge \neg G_1$ tautology law

$= \mathbf{T} \uparrow G_1$ definition of $\uparrow$

The last formula is an S_2 formula.

Case 3: $F = G \wedge H$.

By IH there are S_2 formulas G_1 and $H1$ such that $G \equiv G_1$ and $H_1 \equiv H$, Then

$F = G \wedge H$

$\equiv G_1 \wedge H_1$ IH

$\equiv G_1 \wedge \neg\neg H_1$ double neg intro

$= G_1 \uparrow \neg H_1$ definition of $\uparrow$

$\equiv G_1 \uparrow (\mathbf{T} \uparrow H_1)$ by Case 2

The last formula is S_2.

Grading Criteria: Listing the 3 cases: 3 points

Case 1: 1 point

Case 2: 4 points

Case 3: 4 points

Question 3. (15 points)

Prove or disprove: If $F \vee \neg G$ is satisfiable, then $(F \longrightarrow H) \longrightarrow (G \longrightarrow H)$ is satisfiable. First you must write Proof or Disproof and then provide the proof or the counter-example.

Proof: First we notice 3 things about $\longrightarrow$

(1) $\models_{\mathcal{A}}[J]$ implies $\models_{\mathcal{A}}[I \longrightarrow J]$

(2) $\not\models_{\mathcal{A}}[I]$ implies $\models_{\mathcal{A}}[I \longrightarrow J]$

(3) $[\models_{\mathcal{A}}[I]$ and $\not\models_{\mathcal{A}}[J]]$ implies $\not\models_{\mathcal{A}}[I \longrightarrow J]$

We prove the statement by cases.

Case 1: H is satisfiable. Then it has a model, say $\mathcal{B}$. By (1) we get

(4) $\models_{\mathcal{B}}[G \longrightarrow H]$

We apply (1) again to (4) and get (5).

(5) $\models_{\mathcal{B}}[(F \longrightarrow H) \longrightarrow (G \longrightarrow H)]$

Case 2: G is not a tautology. Then, it has a countermodel $\mathcal{C}$.

By (2) we get (6).

(6) $\models_{\mathcal{C}}[G \longrightarrow H]$

We apply (1) to (6) and get (7).

(7) $\models_{\mathcal{B}}[(F \longrightarrow H) \longrightarrow (G \longrightarrow H)]$

Case 3: H is unsatisfiable and G is a tautology.

Since $F \vee \neg G$ is satisfiable and $\neg G$ and H are unsatisfiable, there is a truth assignment $\mathcal{D}$ such that $\models_{\mathcal{D}}F$ and $\not\models_{\mathcal{D}}H$. By (3) we get (8).

(8) $\not\models_{\mathcal{D}}[F \longrightarrow H]$

Now we apply (2) to (8) and get (9).

(9) $\models_{\mathcal{B}}[(F \longrightarrow H) \longrightarrow (G \longrightarrow H)]$

Since there 3 cases are the only ones possible, we conclude that $(F \longrightarrow H) \longrightarrow (G \longrightarrow H)$ is satisfiable. **Q.E.D.**

Grading Criteria: 1. If you wrote Disproof or Counterexample you get 2 points.

2. If you wrote Proof you get 5 points plus points for the proof.

3. A truth table with headers F, G, H requires a lot of explaining, and by itself is NOT a proof. One way of proving this way it is to state the consequence $(F \vee \neg G) \models ((F \longrightarrow H) \longrightarrow (G \longrightarrow H))$, which reduces to $\models ((F \vee \neg G) \longrightarrow ((F \longrightarrow H) \longrightarrow (G \longrightarrow H))$ by Theorem 1.4.9. Then contruct the truth table of the later formula and show that it is a tautology.

Question 4. (14 points)

Construct a derivation tree of $\square$ from $S = \{\{\neg A, \neg B, C\}, \{A, C, E\}, \{\neg C, D\}, \{\neg C, \neg D\}, \{\neg E, F\}, \{\neg E, \neg F\}, \{B, G\}, \{B, \neg G\}\}$.

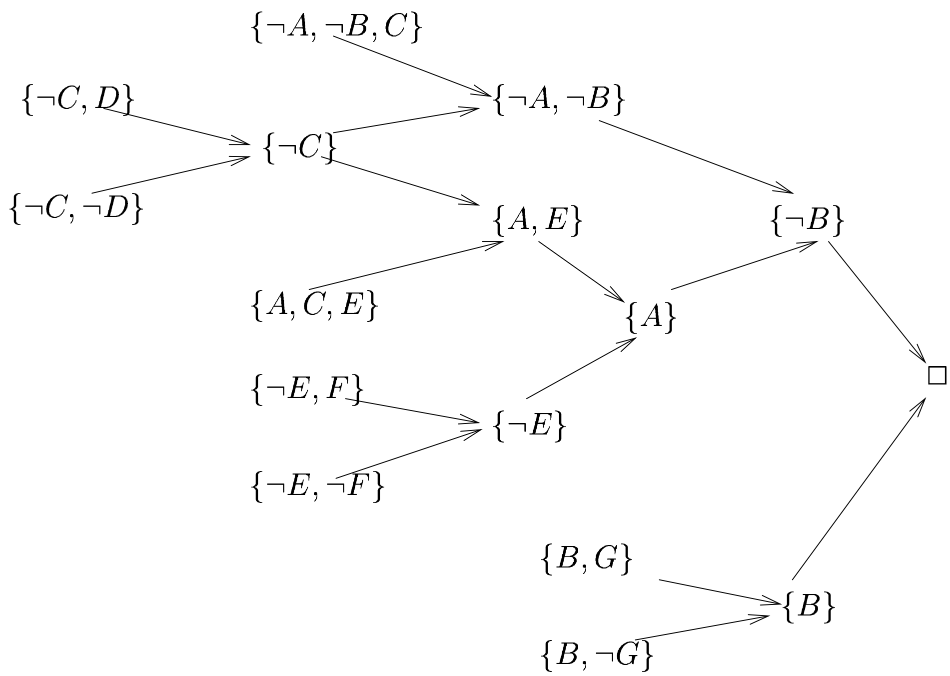

Figure 1: The answer to Question 4

Answer: The tree is shown in Figure 1. You can do better, by finding $\Box$ with only 7 resolution steps.
Grading Criteria: 1. 7/4 points for each correct resolution (up to 8) that leads to $\Box$.

2. -3 points for each incorrect resolution step.

3. -4 points for giving a derivation sequence instead of a tree.

Question 5. (19 points)

Apply the algorithm given in the book to find a CNF for
$F = \neg\{\neg[A \vee (B \vee C)] \longleftrightarrow (\neg C \wedge D)\}$.

Answer:

$F = \neg\{\neg(A \vee B \vee C) \longleftrightarrow (\neg C \wedge D)\}$

$\equiv \neg\{[\neg(A \vee B \vee C) \longrightarrow (\neg C \wedge D)] \wedge [(\neg C \wedge D) \longrightarrow \neg(A \vee B \vee C)]\}$ $\longleftrightarrow$-elim

$\equiv \neg\{[\neg\neg(A \vee B \vee C) \vee (\neg C \wedge D)] \wedge [\neg(\neg C \wedge D) \vee \neg(A \vee B \vee C)]\}$ $\longrightarrow$-elim

$\equiv \neg[\neg\neg(A \vee B \vee C) \vee (\neg C \wedge D)] \vee \neg[\neg(\neg C \wedge D) \vee \neg(A \vee B \vee C)]\}$ DeMorgan's law

$\equiv [\neg\neg\neg(A \vee B \vee C) \wedge \neg(\neg C \wedge D)] \vee [\neg\neg(\neg C \wedge D) \wedge \neg\neg(A \vee B \vee C)]\}$ DeMorgan's law twice

$\equiv [\neg(A \vee B \vee C) \wedge (\neg\neg C \vee \neg D)] \vee [(\neg C \wedge D) \wedge (A \vee B \vee C)]\}$ double neg elim, DeMorgan's, double neg elim 2 times

$\equiv [\neg A \wedge \neg B \wedge \neg C \wedge (C \vee \neg D)] \vee [\neg C \wedge D \wedge (A \vee B \vee C)]\}$ Generalized DeMorgan's, double neg elim

$\equiv (\neg A \vee \neg C) \wedge (\neg A \vee D) \wedge (\neg A \vee A \vee B \vee C) \wedge (\neg B \vee \neg C) \wedge (\neg B \vee D) \wedge (\neg B \vee A \vee B \vee C) \wedge (\neg C \vee \neg C) \wedge (\neg C \vee D) \wedge (\neg C \vee A \vee B \vee C) \wedge (C \vee \neg D \vee \neg C) \wedge (C \vee \neg D \vee D) \wedge (C \vee \neg D \vee A \vee B \vee C)$ Generalized distributivity

$\equiv (\neg A \vee \neg C) \wedge (\neg A \vee D) \wedge (\neg B \vee \neg C) \wedge (\neg B \vee D) \wedge \neg C \wedge (\neg C \vee D) \wedge (A \vee B \vee C \vee \neg D)$ tautology elim, idempotency

$\equiv (\neg A \vee D) \wedge (\neg B \vee D) \wedge \neg C \wedge (A \vee B \vee C \vee \neg D)$ absorbtion

Grading Criteria: 1 point for the first line, 2 points for the other 9.