COT 3420
Section U2
SPRING 2007

EXAM # 1 ANSWERS

QUESTIONS

Question 1.(28 points)

1. a 2. a 3. c 4. a 5. c 6. d 7. b 8. a 9. b 10. d
11. d 12. b 13. c 14. c

Grading Criteria: 2 points for each correct answer.

Question 2. (25 points)

Prove by structural induction that

every non-empty suffix S of F that satisfies the equation $n[(, S] = n[), S]$ is a formula.

Hint: use the lemma that says that every suffix U of a formula G satisfies the inequality $n[), U] \geq n[(, U]$.

Proof: $\mathbf{P}$[F] is

for all strings S, $[(F = US \wedge n[(, S] = n[), S] \wedge S \neq \lambda) \Longrightarrow S \in FORM]$

The implication

$(\dagger)$ $(F = US \wedge n[(, S] = n[), S] \wedge S \neq \lambda) \Longrightarrow S \in FORM$

is true when S is a formula, or when S violates any of the 3 conditions, S is suffix, the counts are equal, or S is non-empty.

We prove $\mathbf{P}[F]$ by induction on F.

Case 1: F is atom. Since atoms are symbols, the only prefixes of F are λ and F.

$S = \lambda$ violates the condition $S \neq \lambda$, so $(\dagger)$ is true. If $S = F$, S is a formula, so $(\dagger)$ is true.

Case 2. $F = \neg G$.

The suffixes of F are the suffixes of G and F.

Subcase 2.1: S is a suffix of G that satisfies the 3 conditions of $(\dagger)$. Then S is formula by IH.

Subcase 2.2. $S = F$. Again, $(\dagger)$ is true.

Case 3: $F = (GCH)$ where C is one of the connectives $\vee, \wedge, \longrightarrow, \longleftrightarrow$.

The sufixes of F fall into 4 categories.
Subcase 3.1: $S = \lambda$. Then (†) is true because S violates the condition $S \neq \lambda$.
Subcase 3.2: $S = I)$ where I is a suffix of H.
Subcase 3.3: $S = JCH)$ with J a suffix of G.
Subcase 3.4: $S = F$

The sufix from 3.1 violates the condition $S \neq \lambda$, and the one in subcase 3.4 is a formula. In both cases, (†) is true. For the other two cases we show that $n[), S] > n[(, S]$.

Subcase 3.2:

$n[), S] = n[), I)]$ $\quad S = I)$
$= n[), I] + 1$
$> n[), I]$ drop the 1
$\geq n[(, I]$ the lemma applied to the suffix I of H
$= n[(, S]$ $\quad S = I)$

So, $n[), S] > n[(, S]$ and (†) is true.

Subcase 3.3:

$n[), S] = n[), JCH)]$ $\quad S = JCH)$
$= n[), J] + n[), H] + 1$
$> n[), J] + n[), H]$ drop the 1
$\geq n[), J] + n[(, H]$ the lemma applied to the suffix H of H
$\geq n[(, J] + n[(, H]$ the lemma applied to the suffix J of G
$= n[(, JCH)]$ $\quad) \neq ($ and $C \neq ($
$= n[(, S]$ $\quad S = JCH)$

So, $n[), S] > n[(, S]$ and (†) is true. **Q.E.D.**

Grading Criteria: List the cases: 3 points

2. Case 1: 2 points
3. Case 2: 6 points.
4. Case 3. 14 points; listing the 4 subcases 3 points, subcases 3.2 and 3.4 4 points each.

Question 3. (15 points)

Proof or disproof: If $F \vee G$ is a tautology, and $\neg G$ is satisfiable, then F is satisfiable.

First you must write if you go for the proof or for disproof. Then, write the proof or display the counterexample on a blank sheet of paper.

Proof: Let $\mathcal{A}$ be a model of $\neg G$. Then, we have (1).

(1) $\mathcal{A}[G] = 0$

Since $F \vee G$ is a tautology, we have (2).

(2) $\mathcal{A}[F \vee G] = 1$

Now,

$\mathcal{A}[F \vee G] = \mathcal{A}[F]\boxed{\vee}\mathcal{A}[G]$ interpretation of $\vee$

$= \mathcal{A}[F]\boxed{\vee}0$ by (1)

$= \mathcal{A}[F]$ table of $\boxed{\vee}$

We have (3).

(3) $\mathcal{A}[F \vee G] = \mathcal{A}[F]$

From (2) and (3) we get (4).

(4)$\mathcal{A}[F] = 1$

So, F is satisfiable. **Q.E.D.**

Grading Criteria: 3 points for writing Disproof or Counterexample.

5 points for writing Proof.

5 points for choosing the correct $\mathcal{A}$.

Question 4. (14 points)

Construct a derivation tree of $\Box$ from

$S = \{\{A, B, C, D\}, \{\neg A, \neg E\}, \{\neg C, F\}, \{\neg A, E\},$
$\{\neg B, D\}, \{\neg D, G\}, \{\neg C, \neg F\}, \{\neg D, \neg G\}\}$.

Answer: The tree is shown in Figure 1.

Grading Criteria: 1. 2 points for each correct resolution (up to 7) that leads to $\Box$.

2. -3 points for each incorrect resolution step.

3. -4 points for giving a derivation sequence instead of a tree.

Question 5. (19 points)

Apply the algorithm given in the book to find a CNF for

$F = \neg[(A \wedge \neg(B \vee C)) \longleftrightarrow \neg(B \vee D)]$.

Solution:

$F = \neg[(A \wedge \neg(B \vee C)) \longleftrightarrow \neg(B \vee D)]$

$= \neg\{[(A \wedge \neg(B \vee C)) \longrightarrow \neg(B \vee D)] \wedge [\neg(B \vee D) \longrightarrow (A \wedge \neg(B \vee C))]\}$ $\longleftrightarrow$-elim

$= \neg\{[\neg(A \wedge \neg(B \vee C)) \vee \neg(B \vee D)] \wedge [\neg\neg(B \vee D) \vee (A \wedge \neg(B \vee C))]\}$ $\longrightarrow$-elim

$= \neg[\neg(A \wedge \neg(B \vee C)) \vee \neg(B \vee D)] \vee \neg[\neg\neg(B \vee D) \vee (A \wedge \neg(B \vee C))]$ De Morgan's law

$= [\neg\neg(A \wedge \neg(B \vee C)) \wedge \neg\neg(B \vee D)] \vee [\neg\neg\neg(B \vee D) \wedge \neg(A \wedge \neg(B \vee C))]$ De Morgan's law twice

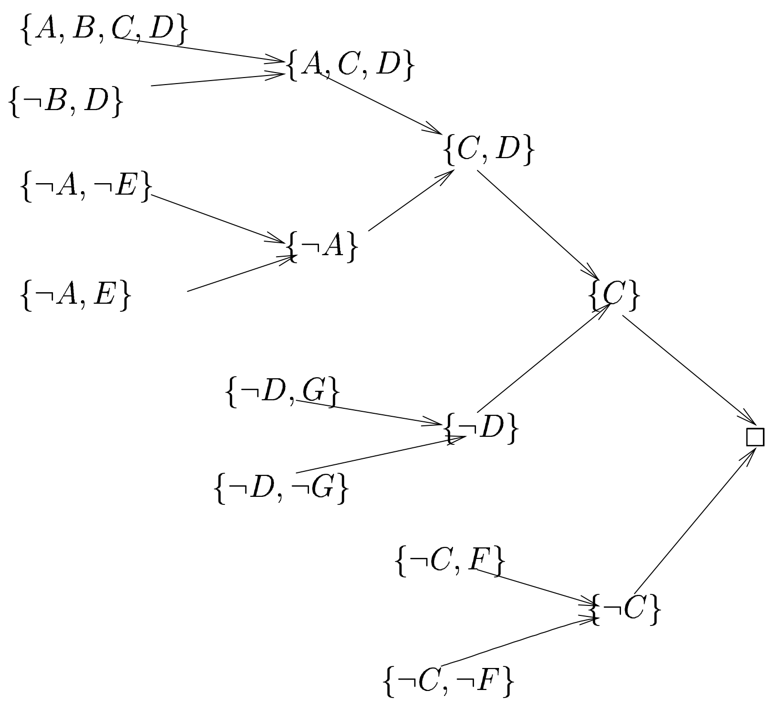

Figure 1: The answer to Question 4

$= [(A \wedge \neg(B \vee C)) \wedge (B \vee D)] \vee [\neg(B \vee D) \wedge (\neg A \vee \neg\neg(B \vee C))]$ double neg elim 3 times, De Morgan's

$= [A \wedge \neg B \wedge \neg C \wedge (B \vee D)] \vee [\neg B \wedge \neg D) \wedge (\neg A \vee B \vee C)]$ double neg elim

$= (A \vee \neg B) \wedge (A \vee \neg D) \wedge (A \vee \neg A \vee B \vee C)$
$\wedge(\neg B \vee \neg B) \wedge (\neg B \vee \neg D) \wedge (\neg B \vee \neg A \vee B \vee C)$
$\wedge(\neg C \vee \neg B) \wedge (\neg C \vee \neg D) \wedge (\neg C \vee \neg A \vee B \vee C)$
$\wedge(B \vee D \vee \neg B) \wedge (B \vee D \vee \neg D) \wedge (B \vee D \vee \neg A \vee B \vee C)$ Generalized distributivity

$= (A \vee \neg B) \wedge (A \vee \neg D) \wedge \neg B \wedge (\neg B \vee \neg D)$
$\wedge(\neg C \vee \neg B) \wedge (\neg C \vee \neg D) \wedge (B \vee D \vee \neg A \vee C)$ tautology elim, idempotency

$= (A \vee \neg D) \wedge \neg B \wedge (\neg C \vee \neg D) \wedge (\neg A \vee B \vee C \vee D)$ absorbtion

Grading Criteria: You get credit up to the first line where you make a mistake or the end of the computation, whatever comes first.

For each line (except the first) you get 2 points, except for line 7 (distributivity) where you get 3 points.