COT 3420
SPRING 2007
Section U1

EXAM # 2 ANSWERS

Question 1. (27 points)

The universe of structure $\mathcal{A}$ is $\{3, 4, 5, 6\}$. The $\mathcal{A}$ interpretations of a, x, and y are $a^{\mathcal{A}} = 3$, $x^{\mathcal{A}} = 5$, $y^{\mathcal{A}} = 6$. The tables for the functions $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ and the predicate $P^{\mathcal{A}}$ are shown in Figure 1. Evaluate the terms and formulas below. We will show the computation, though it was not required.

1. $\mathcal{A}[f(x)] = f^{\mathcal{A}}[x^{\mathcal{A}}] = f^{\mathcal{A}}[5] = 6$

2. $\mathcal{A}[g(x, y)] = g^{\mathcal{A}}[x^{\mathcal{A}}, y^{\mathcal{A}}] = g^{\mathcal{A}}[5, 6] = 6$

3. $\mathcal{A}[g(g(a, y), f(x))] = g^{\mathcal{A}}[g^{\mathcal{A}}[a^{\mathcal{A}}, y^{\mathcal{A}}], f^{\mathcal{A}}[x^{\mathcal{A}}]]$
$= g^{\mathcal{A}}[g^{\mathcal{A}}[3, 6], f^{\mathcal{A}}[5] = g^{\mathcal{A}}[6, 6] = 4$

4. $\mathcal{A}[P(x, f(a))] = P^{\mathcal{A}}[x^{\mathcal{A}}, f^{\mathcal{A}}[a^{\mathcal{A}}]] = P^{\mathcal{A}}[5, f^{\mathcal{A}}[3]]$
$= P^{\mathcal{A}}[5, 4] = 0$

5. $\mathcal{A}[E(f(y), g(x, a))] = 1$ because the terms are mapped into the same element.
$\mathcal{A}[f(y)] = f^{\mathcal{A}}[y^{\mathcal{A}}] = f^{\mathcal{A}}[6] = 3$
$\mathcal{A}[g(x, a)] = g^{\mathcal{A}}[x^{\mathcal{A}}, a^{\mathcal{A}}] = g^{\mathcal{A}}[5, 3] = 3$

6. $\mathcal{A}[\forall x P(x, f(x))] = P^{\mathcal{A}}[3, f^{\mathcal{A}}[3]] \boxed{\wedge} P^{\mathcal{A}}[4, f^{\mathcal{A}}[4]] \boxed{\wedge} P^{\mathcal{A}}[5, f^{\mathcal{A}}[5]] \boxed{\wedge} P^{\mathcal{A}}[6, f^{\mathcal{A}}[6]]$
$= P^{\mathcal{A}}[3, 4] \boxed{\wedge} P^{\mathcal{A}}[4, 5] \boxed{\wedge} P^{\mathcal{A}}[5, 6] \boxed{\wedge} P^{\mathcal{A}}[6, 3] = 1 \boxed{\wedge} 1 \boxed{\wedge} 1 \boxed{\wedge} 1 = 1$

7. $\mathcal{A}[\forall y P(f(y), y)] = P^{\mathcal{A}}[f^{\mathcal{A}}[3], 3] \boxed{\wedge} P^{\mathcal{A}}[f^{\mathcal{A}}[4], 4] \boxed{\wedge} P^{\mathcal{A}}[f^{\mathcal{A}}[5], 5] \boxed{\wedge} P^{\mathcal{A}}[f^{\mathcal{A}}[6], 6]$
$= P^{\mathcal{A}}[4, 3] \boxed{\wedge} P^{\mathcal{A}}[f^{\mathcal{A}}[4], 4] \boxed{\wedge} P^{\mathcal{A}}[f^{\mathcal{A}}[5], 5] \boxed{\wedge} P^{\mathcal{A}}[f^{\mathcal{A}}[6], 6]$
$= 0 \boxed{\wedge} P^{\mathcal{A}}[f^{\mathcal{A}}[4], 4] \boxed{\wedge} P^{\mathcal{A}}[f^{\mathcal{A}}[5], 5] \boxed{\wedge} P^{\mathcal{A}}[f^{\mathcal{A}}[6], 6] = 0$

8. $\mathcal{A}[\forall x \exists y P(y, x)] = [P^{\mathcal{A}}[3, 3] \boxed{\vee} P^{\mathcal{A}}[4, 3] \boxed{\vee} P^{\mathcal{A}}[5, 3] \boxed{\vee} P^{\mathcal{A}}[6, 3]] \boxed{\wedge}$
$[P^{\mathcal{A}}[3, 4] \boxed{\vee} P^{\mathcal{A}}[4, 4] \boxed{\vee} P^{\mathcal{A}}[5, 4] \boxed{\vee} P^{\mathcal{A}}[6, 4]] \boxed{\wedge}$
$[P^{\mathcal{A}}[3, 5] \boxed{\vee} P^{\mathcal{A}}[4, 5] \boxed{\vee} P^{\mathcal{A}}[5, 5] \boxed{\vee} P^{\mathcal{A}}[6, 5]] \boxed{\wedge}$

u	$f^{\mathcal{A}}[u]$
3	4
4	5
5	6
6	3

$f^{\mathcal{A}}[u]$

$u \setminus v$	3	4	5	6
3	3	4	5	6
4	4	4	4	4
5	3	5	5	6
6	3	5	6	4

$g^{\mathcal{A}}[u,v]$

$u \setminus v$	3	4	5	6
3	0	1	0	0
4	0	0	1	0
5	0	0	0	1
6	1	0	0	0

$P^{\mathcal{A}}[u,v]$

Figure 1: Tables for Question 1

$[P^{\mathcal{A}}[3,6]\boxed{\vee}P^{\mathcal{A}}[4,6]\boxed{\vee}\ P^{\mathcal{A}}[5,6]\boxed{\vee}P^{\mathcal{A}}[6,6]]\boxed{\wedge}$
$= 0\boxed{\vee}0\boxed{\vee}\ 0\boxed{\vee}1]\boxed{\wedge}$
$[1\boxed{\vee}P^{\mathcal{A}}[4,4]\boxed{\vee}\ P^{\mathcal{A}}[5,4]\boxed{\vee}P^{\mathcal{A}}[6,4]]\boxed{\wedge}$
$[0\boxed{\vee}1\boxed{\vee}\ P^{\mathcal{A}}[5,5]\boxed{\vee}P^{\mathcal{A}}[6,5]]\boxed{\wedge}$
$[0\boxed{\vee}0\boxed{\vee}\ 1\boxed{\vee}P^{\mathcal{A}}[6,6]]\boxed{\wedge}$
$= 1\boxed{\wedge}1\boxed{\wedge}1\boxed{\wedge}1 = 1$

9. $\mathcal{A}[\exists x \forall y \neg P(x,y)] = [\boxed{\neg}P^{\mathcal{A}}[3,3]\boxed{\wedge}\boxed{\neg}P^{\mathcal{A}}[3,4]\boxed{\wedge}\ \boxed{\neg}P^{\mathcal{A}}[3,5]\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[3,6]]\boxed{\vee}$
$[\boxed{\neg}P^{\mathcal{A}}[4,3]\boxed{\wedge}\boxed{\neg}P^{\mathcal{A}}[4,4]\boxed{\wedge}\ \boxed{\neg}P^{\mathcal{A}}[4,5]\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[4,6]]\boxed{\vee}$
$[\boxed{\neg}P^{\mathcal{A}}[5,3]\boxed{\wedge}\boxed{\neg}P^{\mathcal{A}}[5,4]\boxed{\wedge}\ \boxed{\neg}P^{\mathcal{A}}[5,5]\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[5,6]]\boxed{\vee}$
$[\boxed{\neg}P^{\mathcal{A}}[6,3]\boxed{\wedge}\boxed{\neg}P^{\mathcal{A}}[6,4]\boxed{\wedge}\ \boxed{\neg}P^{\mathcal{A}}[6,5]\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[6,6]]$
$= [\boxed{\neg}0\boxed{\wedge}\boxed{\neg}1\boxed{\wedge}\ \boxed{\neg}P^{\mathcal{A}}[3,5]\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[3,6]]\boxed{\vee}$
$[\boxed{\neg}0\boxed{\wedge}\boxed{\neg}0\boxed{\wedge}\ \boxed{\neg}1\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[4,6]]\boxed{\vee}$
$[\boxed{\neg}0\boxed{\wedge}\boxed{\neg}0\boxed{\wedge}\ \boxed{\neg}0\boxed{\vee}\boxed{\neg}1]\boxed{\vee}$
$[\boxed{\neg}1\boxed{\wedge}\boxed{\neg}P^{\mathcal{A}}[6,4]\boxed{\wedge}\ \boxed{\neg}P^{\mathcal{A}}[6,5]\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[6,6]]$
$= [1\boxed{\wedge}0\boxed{\wedge}\ \boxed{\neg}P^{\mathcal{A}}[3,5]\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[3,6]]\boxed{\vee}$
$[1\boxed{\wedge}1\boxed{\wedge}0\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[4,6]]\boxed{\vee}$
$[1\boxed{\wedge}1\boxed{\wedge}1\boxed{\vee}0]\boxed{\vee}$
$[0\boxed{\wedge}\boxed{\neg}P^{\mathcal{A}}[6,4]\boxed{\wedge}\ \boxed{\neg}P^{\mathcal{A}}[6,5]\boxed{\vee}\boxed{\neg}P^{\mathcal{A}}[6,6]]$
$= 0\boxed{\vee}0\boxed{\vee}0\boxed{\vee}0 = 0$

Grading Criteria: 3 points for each correct answer.

Question 2. (20 points)

Show that the set of connectives $S = \{\forall x(F \wedge \neg G) \wedge \forall y H\}$ is adequate for FOL.
Hint: Let $\sigma[x, F, G, y, H] = \forall x(F \wedge \neg G) \wedge \forall y H$. Find the S meta-formulas $\phi_{\neg}[F]$, $\phi_{\wedge}[F, G]$ and $\phi_{\forall x}[F]$ such that

$\phi_{\neg}[F] \equiv \neg F$, $\phi_{\wedge}[F, G] \equiv (F \wedge G)$ and $\phi_{\forall x}[F] \equiv \forall x F$,

and use the fact that $T = \{\neg F, F \wedge G, \forall x F\}$ is adequate.

Proof:1. We choose $\phi_{\neg}[F] = \sigma[x, E(x,x), F, y, E(y,y)]$ where x is a variable that does not occur in F.

$\sigma[x, E(x,x), F, y, E(y,y)] = \forall x(E(x,x) \wedge \neg F) \wedge \forall y E(y,y)$ definition of σ

$\equiv \forall x(\mathbf{T} \wedge \neg F) \wedge \forall y \mathbf{T}$ tautology law

$\equiv (\mathbf{T} \wedge \neg F) \wedge \mathbf{T}$ eliminate redundant quantifiers

$\equiv \mathbf{T} \wedge \neg F$ tautology law

$\equiv \neg F$ tautology law

2: We chose $\phi_{\wedge}[F, G] = \sigma[x, F, \phi_{\neg}[G], y, E(y,y)]$, where x is a variable that does not occur neither in F nor in G.

$\sigma[x, F, \phi_{\neg}[G], y, E(y,y)] = \forall x(F \wedge \neg\phi_{\neg}[G]) \wedge \forall y E(y,y)$ definition of σ

$\equiv \forall x(F \wedge \neg\phi_{\neg}[G]) \wedge \forall y \mathbf{T}$ $E(y,y) \equiv \mathbf{T}$

$\equiv \forall x(F \wedge \neg\phi_{\neg}[G]) \wedge \mathbf{T}$ eliminate redundant quantifiers

$\equiv \forall x(F \wedge \neg\phi_{\neg}[G])$ tautology law

$\equiv \forall x(F \wedge \neg\neg G)$ part 1

$\equiv \forall x(F \wedge G)$ double negation elim

$\equiv F \wedge G$ eliminate redundant quantifiers

Since $\phi_{\neg}[G]$ is an S meta-formula, so is $\sigma[x, F, \phi_{\neg}[G], y, E(y,y)]$.

3: We choose $\phi_{\forall x}[F] = \sigma[x, E(x,x), \phi_{\neg}[F], x, E(x,x)]$.

$\sigma[x, E(x,x), \phi_{\neg}[F], x, E(x,x)] = \forall x(E(x,x) \wedge \neg\phi_{\neg}[F]) \wedge \forall x E(x,x)$ definition of σ

$\equiv \forall x(\mathbf{T} \wedge \neg\phi_{\neg}[F]) \wedge \forall x \mathbf{T}$ $E(x,x) \equiv \mathbf{T}$

$\equiv \forall x \neg\phi_{\neg}[F] \wedge \forall x \mathbf{T}$ tautology law

$\equiv \forall x \neg\phi_{\neg}[F] \wedge \mathbf{T}$ eliminate redundant quantifiers

$\equiv \forall x \neg\phi_{\neg}[F]$ tautology law

$\equiv \forall x \neg\neg F$ part 1

$\equiv \forall x F$ double negation elim

Since $\phi_{\neg}[F]$ is an S meta-formula, so is $\sigma[x, E(x,x), \phi_{\neg}[F], x, E(x,x)]$.

Grading Criteria:

1. Part 1 : 8 points (5 for choosing ϕ , 3 for the proof)
2. Part 2 : 6 points (4 for choosing ϕ , 2 for the proof)
3. Part 3: 6 points (4 for choosing ϕ , 2 for the proof)

Question 3 (12 points)

Rectify the formula F.

$F = \forall x[\forall x(P(x,y) \vee Q(z,x)) \wedge \forall z(\neg P(z,x) \vee Q(y,z))] \vee \forall y \exists x[\forall x(\neg P(a,x) \vee \neg Q(y,x)) \wedge \exists y(\neg P(y,x) \vee \neg Q(y,b))]$

Answer:

$F \equiv \forall x[\forall x_1(P(x_1,y) \vee Q(z,x_1)) \wedge \forall z_1(\neg P(z_1,x) \vee Q(y,z_1))] \vee \forall y_1 \exists x_2[\forall x_3(\neg P(a,x_3) \vee \neg Q(y_1,x_3)) \wedge \exists y_2(\neg P(y_2,x_2) \vee \neg Q(y_2,b))]$

Grading Criteria: -1 point for each error

Question 4 (8 points)

Skolemize the formula

$F = \exists x \forall y \forall z \exists u \forall v \exists w[(P(x,y) \vee P(a,f(z))) \wedge (\neg P(h(v,x),b) \vee \neg P(u,w))]$.

Don't show your work, just write the Skolemized formula.

Write the substitutions as [variable/term]. Display the answer below.

Answer:

$F \equiv_s \forall y \forall z \forall v[(P(x,y) \vee P(a,f(z))) \wedge (\neg P(h(v,x),b) \vee \neg P(u,w))][x/c][u/g(y,z)][w/i(y,z,v)]$

$= \forall y \forall z \forall v[(P(c,y) \vee P(a,f(z))) \wedge (\neg P(h(v,c),b) \vee \neg P(g(y,z),i(y,z,v)))]$.

Grading Criteria: 1. 2.67 points for each correct substitution.

2. -1.5 points for each overloading.

Question 5 (10 points)

Find a prenex form for

$F = [\exists x(\forall y P(x,y) \vee \forall z Q(x,z)) \wedge \forall u(\exists v \neg P(u,v) \vee \exists w \neg Q(w,u))]$.

Don't show your work, just write the prenex form. Display your answer below.

Answer:

$\exists x \forall u \exists v \exists w \forall y \forall z[(P(x,y) \vee Q(x,z)) \wedge (\neg P(u,v) \vee \neg Q(w,u))]$.

or any quantifier permutation where $\exists x$ precedes both $\forall y$ and $\forall z$ and $\forall u$ precedes both $\exists v$ and $\exists w$.

Grading Criteria: 1. -2.5 points for each precedence violation.

2. -1.5 for each redundant quantifier and quantifier dominated by an $\vee$ or an $\wedge$.

Question 6 (3 points)

Close the formula $F = \exists y \forall v P(x,y,z,u,v)$.

Write your answer below.
Answer: $\exists x \exists z \exists u F$.

Question 7 (20 points)
Let $\forall x \forall y F$ be a rectified formula. Prove that $\forall x \forall y F \models \forall x \forall y F[x/f(x,y), y/g(x,y)]$.
Hint: Apply the relabeling lemma and the translation lemma.
Proof: The key here is to see that the substitution $[x/f(x,y), y/g(x,y)]$ cannot be written as written as a composition of 2 sequential substitutions. So, $[x/f(x,y), y/g(x,y)] \neq [x/f(x,y)][y/g(x,y)]$ and $[x/f(x,y), y/g(x,y)] \neq [y/g(x,y)][x/f(x,y)]$. The reason is that after the first sequential substitution is performed, the second one modifies the term that replaced the first variable. So, $[x/f(x,y)][y/g(x,y)] = [x/f(x,g(x,y)), y/g(x,y)]$ and $[y/g(x,y)][x/f(x,y)] = [y/g(f(x,y),y), x/f(x,y)]$. The Translation Lemma deals with 1 substitution only, so you cannot apply it. You have to relabel $\forall x \forall y F[x/f(x,y), y/g(x,y)]$ and then apply the translation lemma.

So, let u and v be two variables that are not in F.

Then,

$\forall x \forall y F[x/f(x,y), y/g(x,y)] \equiv \forall u \forall y F[x/f(x,y), y/g(x,y)][x/u]$ by the Relabeling Lemma

$= \forall u \forall y F[x/f(u,y), y/g(u,y)]$ $[x/f(x,y), y/g(x,y)][x/u] = [x/f(u,y), y/g(u,y)]$

$\equiv \forall y \forall u F[x/f(u,y), y/g(u,y)]$ swapping universal quantifiers

$\equiv \forall v \forall u F[x/f(u,y), y/g(u,y)][y/v]$ by the Relabeling Lemma

$= \forall v \forall u F[x/f(u,v), y/g(u,v)]$ $[x/f(u,y), y/g(u,y)][y/v] = [x/f(u,v), y/g(u,v)]$

$\equiv \forall u \forall v F[x/f(u,v), y/g(u,v)]$ swapping universal quantifiers

Now the consequence $\forall x \forall y F \models \forall x \forall y F[x/f(x,y), y/g(x,y)]$ becomes

$\forall x \forall y F \models \forall u \forall v F[x/f(u,v), y/g(u,v)]$

and we can apply the Translation Lemma, just like we did in the practice test.

Assume (1).

(1) $\mathcal{A}[\forall x \forall y F]$

Let U be the universe of $\mathcal{A}$. We remove the quantifiers from $\forall x \forall y F$ and get that (2) holds for all $d, e \in U$.

(2) $\mathcal{A}_{[x \leftarrow d][y \leftarrow e]}[F] = 1$

Now we will prove that $\mathcal{A}[\forall u \forall v F[x/f(u,v), y/g(u,v)] = 1$.

$\mathcal{A}[\forall u \forall v F[x/f(u,v), y/g(u,v)] = 1$

iff for all $a \in U$, $\mathcal{A}_{[u \leftarrow a]}[\forall v F[x/f(u,v), y/g(u,v)]] = 1$ interpretation of $\forall u$

iff for all $a \in U$, forall $b \in U$, $\mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[F[x/f(u,v), y/g(u,v)]] = 1$ interpretation of $\forall v$

iff for all $a \in U$, forall $b \in U$, $\mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[F[y/g(u,v)][x/f(u,v)]] = 1$ now $[x/f(u,v), y/g(u,v)] = [y/g(u,v)][x/f(u,v)]$

iff for all $a \in U$, forall $b \in U$, $\mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow \mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[f(u,v)]]}[F[y/g(u,v)]] = 1$ by the Translation Lemma

iff $\mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]]}[F[y/g(u,v)]] = 1$ $\quad \mathcal{A}_{[u \leftarrow a][v \leftarrow b]}[f(u,v)] = f^{\mathcal{A}}[a,b]$

iff $\mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]][y \leftarrow \mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]]}[g(u,v)]]}[F] = 1$ by the Translation Lemma

iff $\mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]][y \leftarrow g^{\mathcal{A}}[a,b]]}[F] = 1$ $\quad \mathcal{A}_{[u \leftarrow a][v \leftarrow b][x \leftarrow f^{\mathcal{A}}[a,b]]}[g(u,v)] = g^{\mathcal{A}}[a,b]$

iff $\mathcal{A}_{[x \leftarrow f^{\mathcal{A}}[a,b]][y \leftarrow g^{\mathcal{A}}[a,b]]}[F] = 1$ $\quad u$ and v are not free in F

The last condition is true because of (2).

Grading Criteria: 1. If you did not relable before you applied The translation lemma you cannot get more than 5 points.