**COT 3420**
**SPRING 2007**
**Section U2**

# EXAM # 2 ANSWERS

**Question 1.** (27 points)

The universe of structure $\mathcal{A}$ is $\{3, 4, 5, 6\}$. The $\mathcal{A}$ interpretations of $a$, $x$, and $y$ are $a^{\mathcal{A}} = 3$, $x^{\mathcal{A}} = 5$, $y^{\mathcal{A}} = 6$. The tables for the functions $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ and the predicate $P^{\mathcal{A}}$ are shown in Figure 1. Evaluate the terms and formulas below. Do not show your work, just write the answer after the equal sign.

1. $\mathcal{A}[f(f(x))] = 3$

2. $\mathcal{A}[g(x, y)] = 3$

3. $\mathcal{A}[g(g(a, y), f(x))] = 5$

4. $\mathcal{A}[P(x, f(a))] = 1$

5. $\mathcal{A}[E(f(y), g(x, a))] = 0$

6. $\mathcal{A}[\forall x P(x, f(f(x)))] = 1$

7. $\mathcal{A}[\forall x(E(f(f(x)), a) \vee E(f(f(x)), f(a)))] = 1$

8. $\mathcal{A}[\forall x \exists y E(g(x, y), a)] = 0$

9. $\mathcal{A}[\exists x \forall y P(y, x)] = 1$

**Grading Criteria:** 3 points for each correct answer.

**Question 2.** (20 points)

Show that the set of connectives $S = \{\forall x(F \wedge \neg G) \vee \forall y H\}$ is not adequate for FOL.

| $u$ | $f^{\mathcal{A}}[u]$ |
|---|---|
| 3 | 4 |
| 4 | 4 |
| 5 | 6 |
| 6 | 3 |

$f^{\mathcal{A}}[u]$

| $u \setminus v$ | 3 | 4 | 5 | 6 |
|---|---|---|---|---|
| 3 | 4 | 5 | 3 | 6 |
| 4 | 4 | 5 | 6 | 4 |
| 5 | 6 | 5 | 4 | 3 |
| 6 | 4 | 4 | 6 | 5 |

$g^{\mathcal{A}}[u, v]$

| $u \setminus v$ | 3 | 4 | 5 | 6 | |
|---|---|---|---|---|---|
| 3 | 0 | 1 | 0 | 0 | |
| 4 | 0 | 1 | 0 | 0 | |
| 5 | 1 | 1 | 0 | 0 | |
| 6 | 0 | 1 | 0 | 0 | |

$P^{\mathcal{A}}[u, v]$

Figure 1: Tables for Question 1

**Answer:** Let us show that the structure that has only one element and assigns one to every atomic formula satisfies every $S$-formula. First, let us define the structure.

$\mathcal{A} =< \{3\}, \mathcal{A}^f, \mathcal{A}^P, \mathcal{A}^v >$ and

1. for all constants $a$, $a^{\mathcal{A}} = 3$,

2. for all functions $g$ of arity $m > 0$, $g^{\mathcal{A}}[3, ..., 3] = 3$, where $[3, 3, ..., 3]$ has $m$ 3's,

3. for all predicate constants $Q$, $Q^{\mathcal{A}} = 1$,

4. for all predicate symbols $Q$ of arity $m > 0$, $Q^{\mathcal{A}}[3, 3, ..., 3] = 1$, where $[3, 3, ..., 3]$ has $m$ 3's,

5. for all variables $x$, $x^{\mathcal{A}} = 3$.

The properties 1, 2, 5 follow from the fact that the universe has only one element.

We have 2 other properties of $\mathcal{A}$.

6. for all terms $t$, $\mathcal{A}[t] = 3$

7. for all formulas $F$, $\mathcal{A}[\forall x F] = \mathcal{A}[\exists x F] = \mathcal{A}[F]$.

Property 6 is true because every term is mapped into an element of the universe, and the universe has only one element.

Let us prove 7.

$\mathcal{A}[\forall x F] = 1$

iff for all $d \in \{3\}$, $\mathcal{A}_{[x \leftarrow d]}[F] = 1$ interpretation of $\forall x$

iff $\mathcal{A}_{[x \leftarrow 3]}[F] = 1$ $d \in \{3\}$ is equivalent to $d = 3$

iff $\mathcal{A}[F] = 1$ $\mathcal{A}_{[x \leftarrow 3]} = \mathcal{A}$ because all variables are mapped into 3.

The proof for $\mathcal{A}[\exists x F] = \mathcal{A}[F]$ is identical, except that for all is replaced by there exists.

Now let us show that $\mathcal{A}$ is a model of all $S$-formulas. The proof is by structural induction on $S$.

Case 1: $F$ is an atom.

Subcase 1.1: $F = Q$, a predicate constant. Then $\mathcal{A}[F] = 1$ by 3.

Subcase 1.2: $F = E(t_1, t_2)$.

By 6, $\mathcal{A}[t_1] = \mathcal{A}[t_2] = 3$, so $\mathcal{A}[F] = 1$.

Subcase 1.3. $F = Q(t_1, ..., t_m)$. Then,

$\mathcal{A}[Q(t_1, ..., t_m)] = Q^{\mathcal{A}}[\mathcal{A}[t_1], ..., \mathcal{A}[t_m]]$ interpretation of $Q(t_1, ..., t_m)$

$= Q^{\mathcal{A}}[3, ..., 3]$ by 6

$= 1$ by 4

Case 2: $F = \forall x(G \wedge \neg H) \vee \forall y I$.

By IH, $\mathcal{A}[H] = 1$. We use 7 and get (8).

(8) $\mathcal{A}[\forall y H] = 1$

(8) implies that $\mathcal{A}[F] = 1$ since $\mathcal{A}$ satisfies one of its disjuncts.

**Question 3** (12 points)

Rectify the formula $F$.

$F = \exists x[\forall y(P(x,y) \vee Q(z,x)) \wedge \forall x \forall z(\neg P(z,x) \vee Q(y,z))] \vee \forall y[\forall x(\neg P(a,x) \vee \neg Q(y,x)) \wedge \exists y(\neg P(y,x) \vee \neg Q(y,b))]$

**Answer:** $F \equiv \exists x_1[\forall y_1(P(x_1,y_1) \vee Q(z,x_1)) \wedge \forall x_2 \forall z_1(\neg P(z_1,x_2) \vee Q(y,z_1))] \vee \forall y_2[\forall x_3(\neg P(a,x_3) \vee \neg Q(y_2,x_3)) \wedge \exists y_3(\neg P(y_3,x) \vee \neg Q(y_3,b))]$

**Grading Criteria:** -1 point for each error.

**Question 4** (8 points)

Skolemize the formula

$F = \forall x \exists y \forall z \exists u \forall v \exists w[((P(x,y) \vee P(a,f(z))) \wedge \neg P(h(v,x),b)) \wedge (P(u,f(v)) \vee \neg P(v,w))]$.

Don't show your work, just write the Skolemized formula.

**Answer:**

$F = \forall x \forall z \forall v F^M[y/g(x)][u/i(x,z)][w/j(x,z,v)]$.

$= \forall x \forall z \forall v[((P(x,g(x)) \vee P(a,f(z))) \wedge \neg P(h(v,x),b)) \wedge (P(i(x,z),f(v)) \vee \neg P(v,j(x,z,v)))]$.

**Grading Criteria:**1. 8/3 points for each correct substitution.

2. -2 points for using a function symbol that is already in the formula.

3. -2 points for overloading a symbol.
4. -1 point for forgetting the universal quantifiers.

**Question 5** (10 points)
Find a prenex form for
$F = [\forall x(\forall y P(x,y) \vee \exists z Q(x,z)) \wedge \forall u(\exists v \neg P(u,v) \vee \exists w \neg Q(w,u))]$.
Don't show your work, just write the prenex form. Display your answer below.

**Answer:** $F \equiv \forall u \exists v \exists w \forall x \exists z \forall y[(P(x,y) \vee Q(x,z)) \wedge (\neg P(u,v) \vee \neg Q(w,u))]$.
In fact every permutation of $\forall u \exists v \exists w \forall x \exists z \forall y$ in which $\forall u$ precedes both $\exists v$ and $\exists w$, and $\forall x$ precedes $\exists z$ is good.

**Grading Criteria:**1. -10/3 points for each precedence violation.
2. -1 point for changing quantifiers.

**Question 6** (3 points)
Find an existential closure of the formula $F = \forall u \exists y P(x,y,z,u,v)$.

**Answer:** $F \equiv_s \exists x \exists z \exists v \forall u \exists y P(x,y,z,u,v)$.

**Grading Criteria:** 1 point for each correct exitential.
2. -1 point for writing $\exists x \exists z \exists v P(x,y,z,u,v)$

**Question 7** (20 points)
Part 1. (8 points)
Show that when $t$ is free for $x$ in $F$, $\forall x F \models F[x/t]$
**Hint:** Apply the translation lemma.
**Answer:**
Let $\mathcal{A}$ be a model of $\forall x F$ and let $U$ be the universe of $\mathcal{A}$. Then, from the interpretation of $\forall x$, we get that (1) is true for all $d \in U$
(1) $\mathcal{A}_{[x \leftarrow d]}[F] = 1$
Since $t$ is free for x in $F$, we apply the Translation Lemma, and get (2).
(2) $\mathcal{A}_{[x \leftarrow \mathcal{A}[t]]}[F] = \mathcal{A}[F[x/t]]$
We set $d = \mathcal{A}[t]$ in (1) and get (3).
(3) $\mathcal{A}_{[x \leftarrow \mathcal{A}[t]]}[F] = 1$
From (2) and (3) we get $\mathcal{A}[F[x/t]]$, i.e. $\mathcal{A}$ is a model of $F[x/t]$.
Part 2. (12 points)

Show that when $t$ is not free for $x$ in $F$, $\forall x F \models F[x/t]$ is not always true.
**Hint:** Construct a counter-example.
**Counter-example:** Let $F = \exists y \neg E(x,y)$, $t = y$, and $\mathcal{A}$ a structure with universe $\{3,4\}$. Then,

$\mathcal{A}[\forall x \exists y \neg E(x,y)]$
$= \mathcal{A}_{[x\leftarrow 3]}[\exists y \neg E(x,y)] \boxed{\wedge} \mathcal{A}_{[x\leftarrow 4]}[\exists y \neg E(x,y)$ interpretation of $\forall x$
$= [\mathcal{A}_{[x\leftarrow 3][y\leftarrow 3]}[\neg E(x,y)] \boxed{\vee} \mathcal{A}_{[x\leftarrow 3][y\leftarrow 4]}[\neg E(x,y)]]$
$\boxed{\wedge} [\mathcal{A}_{[x\leftarrow 4][y\leftarrow 3]}[\neg E(x,y)] \boxed{\vee} \mathcal{A}_{[x\leftarrow 4][y\leftarrow 4]}[\neg E(x,y)]]$ interpretation of $\exists y$
$= [\boxed{\neg} 3 = 3 \boxed{\vee} \boxed{\neg} 3 = 4] \boxed{\wedge} [\boxed{\neg} 4{=}3 \boxed{\vee} \boxed{\neg} 4{=}4]$ interpretation of of $\neg$ and $E$
$= 1 \boxed{\wedge} 1$
$= 1$

At the same time,
$\mathcal{A}[F[x/t]]$
$= \mathcal{A}[\exists y \neg E(y,y)]$ $F = \exists y \neg E(x,y)$, $t = y$
$= \mathcal{A}[\exists y \neg \mathbf{T}]$ $E(y,y)$ is a tautology
$= \mathcal{A}[\exists y \Box]$ contradiction law
$= \mathcal{A}[\Box]$ eliminate redundant quantifiers
$= 0$ interpretation of $\Box$