COT 3420
Sections U1
SPRING 2007

FINAL EXAM ANSWERS

QUESTIONS

Question 1. (25 points)

Let C be a clause and $L_1, L_2, \ldots L_n$, $n \geq 2$, be a set of literals of C such that $S = \{L_1, \ldots, L_n\}$ is unifiable. Let σ be a mgu of S. The clause $F = \sigma[C]$ is called a **factor** of C.

For example, let $C = \{P(x, y), P(y, z), Q(x, y, z)\}$ and $S = \{P(x, y), P(y, z)\}$. S is unifiable and $\sigma = [y/x, z/x]$ is an mgu of S. Then, the clause $R = \sigma[C] = \{P(x, x), Q(x, x, x)\}$ is a factor of C.

Now, let C be a clause, R a factor of C, and $\overline{C}$ and $\overline{R}$ be the universal closure of C and R.

Prove that $\overline{C} \models \overline{R}$.

Proof: Let $\sigma = [x_1/t_1, \ldots, x_n/t_n]$ and $y_1, \ldots, y_m$ be the other variables of C. Then

(1) $\overline{C} = \forall y_1 \ldots \forall y_m \forall x_1 \ldots \forall x_n C$

Let $V = u_1, \ldots u_p$ be the set of variables in R, and $z_1, \ldots z_p$ be a set of variables that do not occur neither in C nor in R. Let $r = [u_1/z_1, \ldots, u_p/z_p]$.

Then,

(2) $\overline{R} = \forall u_1 \ldots \forall u_p R$

By the relabeling lemma, (2) implies (3).

(3) $\overline{R} \equiv \forall z_1 \ldots \ldots \forall z_p r[R]$.

Now, $r[R] = r[\sigma[C]] = [r \diamond \sigma][C]$ and

$[r \diamond \sigma][C] = [x_1/\tau_1, \ldots, x_n/\tau_n, y_1/\tau_{n+1}, \ldots, y_m/\tau_{n+m}][C]$ where $\tau_i = \overline{r}[t_i]$ for $1 \leq i \leq n$ and $\tau_{n+j} = r[y_j]$ for $1 \leq j \leq m$. The variables of the τ terms are in the set $z_1, \ldots z_p$.

So,

(4) $\overline{R} \equiv \forall z_1 \ldots \forall z_p C[x_n/\tau_n] \ldots [x_1/\tau_1][y_m/\tau_{n+m}] \ldots [y_1/\tau_{n+1}]$

We have to show that (5) holds for all structures $\mathcal{A}$.

(5) $\mathcal{A}[\overline{C}] = 1$ implies $\mathcal{A}[\forall z_1 \ldots \forall z_p C[x_n/\tau_n] \ldots [x_1/\tau_1][y_m/\tau_{n+m}] \ldots [y_1/\tau_{n+1}] = 1$

This is done by the Translation Lemma. We assume (6).

(6) $\mathcal{A}[\overline{C}] = 1$

Let now $d_1, \ldots, d_p \in U$ and $\mathcal{B} = \mathcal{A}_{[z_1 \leftarrow d_1]\ldots[z_p \leftarrow d_p]}$.

$\mathcal{B}[C[x_n/\tau_n]\ldots[x_1/\tau_1][y_m/\tau_{n+m}]\ldots[y_1/\tau_{n+1}]]$

$= \mathcal{B}_{[y_1 \leftarrow \mathcal{B}[\tau_{n+1}]]}[C[x_n/\tau_n]\ldots[x_1/\tau_1][y_m/\tau_{n+m}]\ldots[y_2/\tau_{n+2}]]$ by the translation lemma

$= \mathcal{B}_{[y_1 \leftarrow \mathcal{B}[\tau_{n+1}]][y_2 \leftarrow \mathcal{B}_{[y_1 \leftarrow \mathcal{B}[\tau_{n+1}]]}[\tau_{n+2}]]}[C[x_n/\tau_n]\ldots[x_1/\tau_1][y_m/\tau_{n+m}]\ldots[y_3/\tau_{n+3}]]$ by the translation lemma

$= \mathcal{B}_{[y_1 \leftarrow \mathcal{B}[\tau_{n+1}]][y_2 \leftarrow \mathcal{B}[\tau_{n+2}]]}[C[x_n/\tau_n]\ldots[x_1/\tau_1][y_m/\tau_m]\ldots[y_3/\tau_3]]$ y_1 does not occur in τ_{n+2}

$= \ldots$

$= \mathcal{B}_{[y_1 \leftarrow \mathcal{B}[\tau_{n+1}]][y_2 \leftarrow \mathcal{B}[\tau_{n+2}]]\ldots[y_m \leftarrow \mathcal{B}[\tau_{n+m}]]\ldots[x_1 \leftarrow \mathcal{B}[\tau_1]]\ldots[x_n \leftarrow \mathcal{B}[\tau_n]]}[C]$ repeated applications of the translation lemma.

$= \mathcal{A}_{[y_1 \leftarrow \mathcal{B}[\tau_{n+1}]][y_2 \leftarrow \mathcal{B}[\tau_{n+2}]]\ldots[y_m \leftarrow \mathcal{B}[\tau_{n+m}]]\ldots[x_1 \leftarrow \mathcal{B}[\tau_1]]\ldots[x_n \leftarrow \mathcal{B}[\tau_n]]}[C]$ $z_1, \ldots z_p$ do not occur in C

$= 1$ by (6)

Question 2. (20 points)

Write the predicate partition(Pivot,List,List1,List2) that splits List into List1 and List2 in such a way that List1 contains the elements of List that are less than or equal to Pivot and List2 contains the items greater than Pivot. For example partition(12,[3, 20, 23, 5, 12, 18, 4 ,6], L1,L2) is satisfied if L1 = [3, 5, 12, 4, 6] and L2 = [20, 23, 18].

```
% partition(Pivot,List,List1,List2) that splits List into List1 and List2 in such a way
% that List1 contains the elements of List that are less than or equal to Pivot
% and List2 contains the items greater than Pivot.
partition( _,[],[],[]). % the empty list generates 2 empty lists
partition(Pivot, [Head| Tail], [Head | Tail1], List2):-
    (Head < Pivot ; Head = Pivot), // put Head list 2
    partition(Pivot, Tail, Tail1, List2).
partition(Pivot, [Head| Tail], List1 , [Head | Tail2]):-
    partition(Pivot, Tail, List1 , Tail2).
```

Grading Criteria:

1. 4 points for clause 1

2. 8 points for clause 2
3. 6 points for clause 3
4. 2 points for style (comments, good variable names, indentation)

Question 3. (20 points)

Write $D[F,2]$ for $F = \forall x \forall y [P(f(x,y),y) \wedge (\neg P(x,g(x,y)) \vee \neg P(f(x,x),y))]$.
Write your answer below.

Answer $D[F,2] = \{a, f(a,a), g(a,a), f(a,f(a,a)), f(a,g(a,a)), f(f(a,a),a), f(f(a,a),f(a,a)),$
$f(f(a,a),g(a,a)), f(g(a,a),a), f(g(a,a),f(a,a)), f(g(a,a),g(a,a)), g(a,f(a,a)),$
$g(a,g(a,a)), g(f(a,a),a), g(f(a,a),f(a,a)), g(f(a,a),g(a,a)), g(g(a,a),a),$
$g(g(a,a),f(a,a)), g(g(a,a),g(a,a))\}$

Grading Criteria: 1 bonus point + 1 point for each right term.
2. -1 point for each term that is not in $D[F,2]$

Question 4. (10 points)

Write all resolvents of the clauses $C_1 = \{P(x,f(x),y), P(g(y),z,h(u)), Q(x,y,z)\}$ and $C_2 = \{\neg P(x,y,z), Q(x,z,y)\}$. Don't write the relabelings, the unification set, and the mgu, just display the resolvents. Write your answer below.

Answer:

$R_1 = \{Q(g(h(u)), h(u), f(g(h(u))))\}$, $R_2 = \{P(g(y),z,h(u)), Q(x,y,z), Q(x,y,f(x))\}$,
$R_3 = \{P(x,f(x),y), Q(x,y,z), Q(g(y),h(u),z)\}$

1. 4 points for R_1, 3 points for each of R_2 and R_3.
2. -1 point for each wrong term.

Question 5. (10 points)

Find out if $S = \{P(f(x),y,h(x,z)), P(y,f(g(u)),v), P(f(g(a)),f(w),h(w,f(w)))\}$ is unifiable. If so, display an mgu; otherwise write that S is not unifiable.
Write your answer below.

Answer: $\sigma = [u/a, v/h(g(a),f(g(a))), w/g(a), x/g(a), y/f(g(a)), z/f(g(a))]$.

Grading Criteria: 10/6 points for each correct pair.

Question 6. (30 points)

For each of the following statements select the string that best completes its meaning. There is no penalty for wrong guessing, but choose only one answer.

1. $[x/z, y/x, v/w] \diamond [x/y, y/u, u/x, w/v] = \ldots$
   a. $[y/u, u/z]$
   b. $[x/z, u/x, w/v]$
   c. $[y/u, u/z, v/w]$
   d. $[x/z, y/u, u/z, v/w]$
2. $[v/b] \diamond [z/h(u)] \diamond [x/g(v)] \diamond [y/f(x,z)] = \ldots$
   a. $[y/f(g(v), z), x/g(v), z/h(u), v/b]$
   b. $[y/f(g(v), h(u)), x/g(b), z/h(b), v/b]$
   c. $[y/f(g(b), h(u)), x/g(v), z/h(u), v/b]$
   d. $[y/f(g(b), h(u)), x/g(b), z/h(u), v/b]$
3. ...is unifiable.
   a. $\{P(a, x), P(x, b)\}$
   b. $\{P(x, f(y)), P(x, a)\}$
   c. $\{P(x, y), P(f(x), z)\}$.
   d. $\{P(f(x), x), P(y, g(z))\}$.
4. ...is not a tautology.
   a. $\exists x \forall x F \longrightarrow \forall x F$
   b. $\forall x(F \wedge G) \longrightarrow (\forall x F \wedge \forall x G)$
   c. $(\exists x F \wedge \exists x G) \longrightarrow \exists x(F \wedge G)$
   d. $F[x/a] \longrightarrow \exists x F$
5. ...is an mgu of $\{P(f(x, y), y), P(z, h(x)), P(f(g(u), y), h(v))\}$
   a. $\sigma = [x/g(u), y/h(g(u)), z/f(g(u), h(g(u)), v/u]$
   b. $\sigma = [x/g(u), y/h(g(u)), z/f(g(u), y), v/g(u)]$
   c. $\sigma = [x/g(u), y/h(x), z/f(g(u), h(g(u))), v/u]$
   d. $\sigma = [x/g(u), y/h(g(u)), z/f(g(u), h(g(u))), v/g(u)]$
6. The substitution ...is a unifier of $S = \{P(x, f(x)), P(g(y), f(z))\}$.
   a. $[x/g(a), y/a, z/g(a)]$
   b. $[x/g(a), y/a, z/x]$
   c. $[x/g(y), z/x]$
   d. $[x/g(a), y/a, z/a]$
7. Let s be a substitution such that $s \diamond s = []$. Then ...
   a. s is a relabeling.
   b. $s = []$
   c. $x/y \in s$ if and only if $y/x \in s$.
8. ...is a rename (relabeling).
   a. [x/a, y/x]
   b. $[x/y, y/x]$
   c. $[x/y, y/z, z/y]$

9. Let S be a finite set of FOL clauses. Then, $Res^*[S]$ ...
   a. is finite.
   b. is infinite.
   c. is countable.
10. ...does not preserve the relation $\equiv_s$.
   a. $\exists x$
   b. $\vee$
   c. $\neg$

Answers: 1. c 2. d 3. d 4. c 5. d 6. a 7. c 8. b 9. c 10. c

Grading Criteria: 3 points for each correct answer.

Question 7. (15 points)

Prove by first order resolution that the set $S = \{\{P(f(x), y), P(y, f(z)), Q(x, y, f(x))\}, \{\neg P(x, y), Q(z, x, y)\}, \{\neg R(g(x), f(y)), \neg R(g(z), u), \neg Q(x, u, f(z))\}, \{R(x, y), \neg Q(z, y, y)\}\}$ is unsatisfiable. For each resolution step specify the relabelings, the unification set and the mgu. Do the minimal number of unifications.

Answer: The tree is shown in Figure 1.

Grading Criteria: 1. 5 points for each correct resolution step leading to $\square$.

2. For each step, 0.5 points for relabelings, 0.5 points for S, 1.5 points for mgu and 2.5 points for the resolvent.

3. If both parents of a resolvent are wrong, no credit is given for the step.

4. If one of the parent clauses is wrong you cannot get more than 3 points for the resolvent, even if it is correct.

5. -2 points for each extra resolution step.

Question 8. (16 points)

We create the data base shown below.

```
arc(b,c).
arc(c,a).
arc(a,b).
arc(a,d).
path1(X,X).
path1(X,Y) :- path1(X,Z), arc(Z,Y).
```

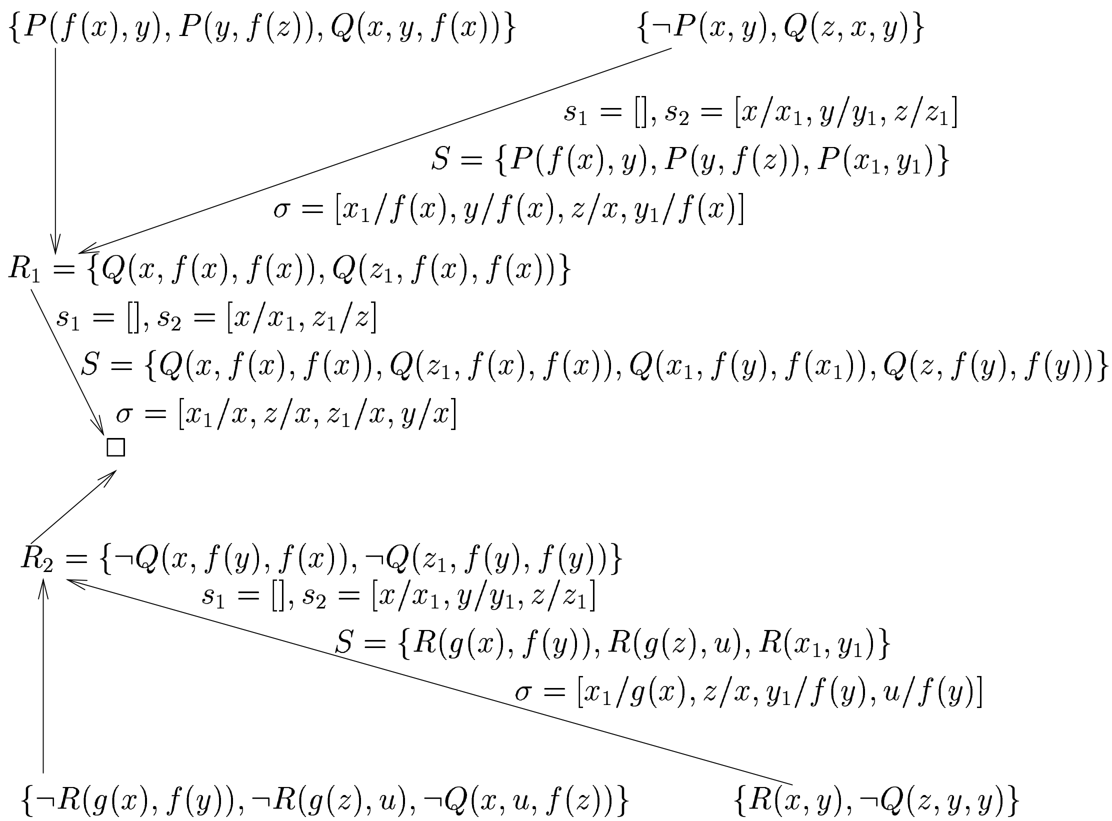

Figure 1: The answer to Question 7

path2(X,X).
path2(X,Y) :- arc(X,Z), path2(Z,Y).
What will be printed out by the queries below?
Write yes, no, or out of local stack next to the query.

?- path1(a,d). yes

?- path2(a,d). out of local stack

?- path1(d,a). out of local stack

?- path2(d,a). no

Grading Criteria: 4 points for each correct answer.

Question 9. (10 points)

Let σ be an mgu of $S = \{A_1, \ldots A_n, B_1, \ldots B_m\}$. Prove that $T_1 = \{A_1, \ldots A_n\}$ and $T_2 = \{B_1, \ldots B_m\}$ have mgu's.

Solution: T_1 and T_2 are subsets of S, so σ is a unifier for them. By the unification theorem, T_1 and T_2 have mgu's.

Question 10 (20 points)

Let $S = \{A_1, \ldots, A_n\}$ be a unifiable set of atoms, Var be the set of variables that occur in S, and σ be an mgu algorithm produced by the mgu algorithm from the book. Prove that $\sigma = [x_1/t_1, \ldots, x_n/t_n]$ where $x_1, \ldots, x_n \in Var$ and $t_1, \ldots t_n$ have variables in $Var - \{x_1, \ldots, x_n\}$.

Proof: We prove by induction on m that σ_m satisfies the property †.

(†) $\sigma_m = [x_1/t_{m,1} \ldots, x_m/t_{m,m}]$, $x_1, \ldots, x_m \in Var$ and $t_{m,1}, \ldots, t_{m,m}$ do not contain $x_1, \ldots, x_m$.

Basis: $m = 0$. Since $\sigma_0 = []$ the statement is vacuously true.

Inductive Step: Assume that σ_m satisfies (†). If the disagreement set of S_m is empty, σ_m is the mgu and we are done. If not, we find a pair $x_{m+1}/t_{m+1,m+1}$ such that $t_{m+1,m+1}$ does not contain x_{m+1}. If we cannot find such a pair, the set S is not unifiable, contrary to our assumption.

$S_m = \sigma_m[S]$ does not contain any of the variable $x_1, \ldots, x_m$ because they do not occur in $t_{m,1}, \ldots, t_{m,m}$. Since $t_{m+1,m+1}$ is in S_m it doesn't have $x_1, \ldots, x_m$ occurrences. By choice, it does not have x_{m+1}. Now,

$\sigma_{m+1} = [x_{m+1}/t_{m+1,m+1}] \diamond \sigma_m$
$= [x_1/t_{1,m}[x_{m+1}/t_{m+1,m+1}], \ldots, x_m/t_{m,m}[x_{m+1}/t_{m+1,m+1}], x_{m+1}/t_{m+1,m+1}]$

The terms $t_{i,m+1} = t_{i,m}[x_{m+1}/t_{m+1,m+1}]$, $1 \leq i \leq m$, don't have any $x_1, \ldots, x_m$ and we replace x_{m+1} by a term that doesn't contain $x_1, \ldots, x_{m+1}$. Since $t_{m+1,m+1}$ satisfies the same condition, σ_{m+1} fulfills †.

Grading Criteria: 1. Trying to prove the statement by induction : 6 points

2. Stating † : 5 points
3. Basis : 1 point
4. The inductive step: 8 points