COT 3420
Sections U2
SPRING 2007

FINAL EXAM ANSWERS

QUESTIONS

Question 1. (25 points)

Prove the Subsumption Theorem for FOL. Let C_1 and C_2 be two clauses such that $C_1 \subseteq C_2$. Let $\overline{C_1}$ and $\overline{C_2}$ be the universal closures of C_1 and C_2 in this order. Prove that $\overline{C_1} \wedge \overline{C_2} \equiv \overline{C_1}$.

Proof: We need to show that $(\overline{C_1} \wedge \overline{C_2}) \models \overline{C_1}$ and $\overline{C_1} \models (\overline{C_1} \wedge \overline{C_2})$. The first consequence is automatic since the consequent is a subset of the assumptions. For the second we need to prove that $\overline{C_1} \models \overline{C_2}$.

Since $C_1 \subseteq C_2$, all variables of C_1 occur in C_2. Let $x_1, \ldots, x_n$ be the variables of C_1 and $y_1, \ldots, y_m$ be the extra variables of C_2.

So, we get (1) and (2).

(1) $\overline{C_1} = \forall x_1 \ldots \forall x_n C_1$

(2) $\overline{C_2} = \forall x_1 \ldots \forall x_n \forall y_1 \ldots \forall y_m C_1$

Now let $\mathcal{A}$ be a structure that satisfies $\overline{C_1}$ and U the universe of $\mathcal{A}$

Then, (3) holds for all $d_1, \ldots, d_n \in U$.

(3) $\mathcal{A}_{[x_1 \leftarrow d_1] \ldots [x_n \leftarrow d_n]}[C_1] = 1$

Now let $e_1, \ldots, e_{n+m} \in U$. We need to show (4).

(4) $\mathcal{A}_{[x_1 \leftarrow e_1] \ldots [x_n \leftarrow e_n][y_1 \leftarrow e_{n+1}] \ldots [y_m \leftarrow e_{n+m}]}[C_2] = 1$

By (3), $\mathcal{A}_{[x_1 \leftarrow d_1] \ldots [x_n \leftarrow e_n]}[C_1] = 1$. Since, C_1 is a disjunction of literals, there is a literal $L \in C_1$ such that (5) holds.

(5) $\mathcal{A}_{[x_1 \leftarrow e_1] \ldots [x_n \leftarrow e_n]}[L] = 1$

The variables of L are in the set $x_1, \ldots, x_n$, so $\mathcal{A}_{[x_1 \leftarrow e_1] \ldots [x_n \leftarrow e_n]}$ and $\mathcal{A}_{[x_1 \leftarrow e_1] \ldots [x_n \leftarrow e_n][y_1 \leftarrow e_{n+1}] \ldots [y_m \leftarrow e_{n+m}]}$ agree on L.

Hence, we have (6).

$\mathcal{A}_{[x_1 \leftarrow e_1] \ldots [x_n \leftarrow e_n][y_1 \leftarrow e_{n+1}] \ldots [y_m \leftarrow e_{n+m}]}[L] = 1$

Since $C_1 \subseteq C_2$, $L \in C_2$ and (4) is true.

The literal L may vary according to the assignments $[x_1 \leftarrow e_1] \ldots [x_n \leftarrow e_n]$, but we will always find at least one such L. Hence, (4) is true for all assignments $[x_1 \leftarrow e_1] \ldots [x_n \leftarrow e_n][y_1 \leftarrow e_{n+1}] \ldots [y_m \leftarrow e_{n+m}]$

Q.E.D.

Question 2. (20 points)

Write a prolog predicate count(List,Item,Sum) which is satisfied if List has Sum occurrences of Item. For example, count([a, 3, a, 6, a, 3, a], a, Total) is satisfied if Total = 4.

% count (List,Item,Sum) is satisfied when List has Sum occurrences of Item.
count([],_ ,0). % the empty list has 0 items of any kind
count([Item | Tail], Item, N) :- count(Tail, Item, M), % Tail has N items
N is M + 1. % the list has one more
count([Head | Tail], Item, N) :- count(Tail, Item, N). % item is not at the head

Grading Criteria: 1. First clause : 5 points
2. Second clause : 8 points
3. Third clause : 7 points

Question 3. (20 points)

Write $E[F,2]$ for $F = \forall x \forall y \forall z[(P(x, f(x)) \vee P(f(y), a)) \wedge (\neg P(y, f(z)) \vee \neg P(a, z))]$. You can write F^M for $[(P(x, f(x)) \vee P(f(y), a)) \wedge (\neg P(y, f(z)) \vee \neg P(a, z))]$.

Answer: $D[F,2] = \{a, f(a), f^2(a)\}$.

$E[F,2] = \{F^M[x/t_1, y/t_2, z/t_3] | t_1, t_2, t_3 \in D[F,2]\}$
$= \{F^M[x/a, y/a, z/a], F^M[x/a, y/a, z/f(a)], F^M[x/a, y/a, z/f^2(a)], F^M[x/a, y/f(a), z/a],$
$F^M[x/a, y/f(a), z/f(a)], F^M[x/a, y/f(a), z/f^2(a)], F^M[x/a, y/f^2(a), z/a],$
$F^M[x/a, y/f^2(a), z/f(a)], F^M[x/a, y/f^2(a), z/f^2(a)], F^M[x/f(a), y/a, z/a],$
$F^M[x/f(a), y/a, z/f(a)], F^M[x/f(a), y/a, z/f^2(a)], F^M[x/f(a), y/f(a), z/a],$
$F^M[x/f(a), y/f(a), z/f(a)], F^M[x/f(a), y/f(a), z/f^2(a)], F^M[x/f(a), y/f^2(a), z/a],$
$F^M[x/f(a), y/f^2(a), z/f(a)], F^M[x/f(a), y/f^2(a), z/f^2(a)], F^M[x/f^2(a), y/a, z/a],$
$F^M[x/f^{(}a), y/a, z/f(a)], F^M[x/f^2(a), y/a, z/f^2(a)], F^M[x/f^2(a), y/f(a), z/a],$
$F^M[x/f^2(a), y/f(a), z/f(a)], F^M[x/f^2(a), y/f(a), z/f^2(a)], F^M[x/f^2(a), y/f^2(a), z/a],$
$F^M[x/f^2(a), y/f^2(a), z/f(a)], F^M[x/f^2(a), y/f^2(a), z/f^2(a)]$

Grading Criteria: 20/27 points for each right member; -20/27 points for each formula that is not in $E[F,2]$.

Question 4. (10 points)

Write all resolvents of the clauses $C_1 = \{P(x, g(y)), P(f(z), z), Q(x, y, z)\}$ and $C_2 = \{\neg P(f(x), y), Q(f(y), z, x)\}$. Don't write the relabelings, the unification set, and the mgu, just display the resolvents. Write your answer below.

Answer: $R_1 = \{Q(f(g(y)), y, g(y)), Q(f(g(y)), z_1, g(y))\}$,
$R_2 = \{P(f(z), z), Q(f(x_1), y, z), Q(f(g(y)), z_1, x_1)\}$,
$R_3 = \{P(x, g(y)), Q(x, y, z), Q(f(z), z_1, z)\}$.

Grading Criteria: 1. 4 points for R_1, 3 points for each of R_2, R_3.
2. -1 points for each wrong term.

Question 5. (10 points)

Find out if $S = \{P(x, f(y), g(x, y)), P(h(y), z, g(h(u), f(v))), P(h(u), f(f(v)), w)\}$ is unifiable. If so, display a mgu; otherwise write that S is not unifiable. Write your answer below.

Answer: $\sigma = [x/h(f(v)), y/f(v), z/f^2(v), u/f(v), w/g(h(f(v)), f(v))]$.

Grading Criteria: 2 points for each correct member of σ.

Question 6. (30 points)

For each of the following statements select the string that best completes its meaning. There is no penalty for wrong guessing, but choose only one answer.

1. $[x/z, z/x, v/u, w/v] \diamond [x/y, y/z, z/x, u/v] = \ldots$
   a. $[x/y, y/x, w/v]$
   b. $[z/y, w/v, y/z, u/v]$
   c. $[z/y, w/v, y/z]$
   d. $[x/y, y/x, v/u, w/v]$
2. $[v/b] \diamond [y/h(z)] \diamond [u/g(y, v)] \diamond [x/f(y, z)] = \ldots$
   a. $[x/f(h(z), z), u/g(h(z), v), y/h(z), v/b]$
   b. $[x/f(h(z), z), u/g(y, v), y/h(z), v/b]$
   c. $[x/f(h(z), z), u/g(h(z), b), y/h(z), v/b]$
   d. $[x/f(h(z), z), u/g(h(z), b), y/h(b), v/b]$
3. $\ldots$ is unifiable.
   a. $\{P(x, y), P(f(x), z)\}$
   b. $\{P(x, f(y)), P(y, a)\}$
   c. $\{P(y, f(z)), P(f(x), y)\}$.
   d. $\{P(f(x), x), P(y, y)\}$.
4. $\ldots$ is not a tautology.
   a. $\exists x P(x) \longrightarrow P(x)$
   b. $P(x, y) \longrightarrow \exists x \exists y P(x, y)$
   c. $\forall x P(x) \longrightarrow \exists x P(x)$

d. $\forall x \forall y P(x,y) \longrightarrow \forall x P(x,x)$

5. ... is an mgu of $\{P(f(x,y),y), P(z,h(x)), P(f(g(u),y),h(v))\}$

a. $\sigma = [x/g(u), y/h(g(u)), z/f(g(u),h(g(u)), v/u]$

b. $\sigma = [x/g(u), y/h(g(u)), z/f(g(u),y), v/g(u)]$

c. $\sigma = [x/g(u), y/h(x), z/f(g(u),h(g(u))), v/u]$

d. $\sigma = [x/g(u), y/h(g(u)), z/f(g(u),h(g(u))), v/g(u)]$

6. The substitution ... is a unifier of $S = \{P(x,f(x),y), P(g(y),z,h(u))\}$.

a. $[x/g(y), y/h(u), z/f(g(h(u)))]$

b. $[x/g(a), y/h(a), z/f(g(h(a)))]$

c. $[x/g(h(a)), y/h(a), z/f(g(h(a))), u/a]$

d. $[x/g(h(a)), y/h(h(a)), z/f(g(h(a)))]$

7. Let s and σ be two substitution such that $s \diamond \sigma = []$. Then ...

a. s is a relabeling.

b. $s = \sigma = []$

c. $x/y \in s$ if and only if $y/x \in \sigma$.

8. ... is a ground substitution.

a. [x/a, y/x]

b. $[x/y, y/x]$

c. $[x/y, y/z, z/y]$

d. []

9. Let us define the relation $\uparrow$ on the set of atoms as $P \uparrow Q$ if P and Q are unifiable. Then, $\uparrow$ is ...

a. symmetric.

b. antisymmetric.

c. transitive.

10. The lifting lemma tells us that ...

a. we can lift derivation trees from the Herbrand expansions to FOL.

b. the connectives $\neg, \vee, \wedge, \longrightarrow, \longleftrightarrow, \forall x, \forall y$ preserve equivalences.

c. whenever t is free for x in F, $\mathcal{A}_{[x \leftarrow \mathcal{A}[t]]}[F] = \mathcal{A}[F]$.

d. a set of FOL clauses S is unsatisfiable iff it has a finite subset that is unsatisfiable.

Answers: 1. d 2. c 3. c 4. a 5. d 6. c 7. c 8. d 9. a 10. a

Grading Criteria: 3 points for each correct answer.

Question 7. (15 points)

Prove by first order resolution that the set $S = \{\{P(x,f(x)), P(g(y),z), Q(x,y,z)\},$

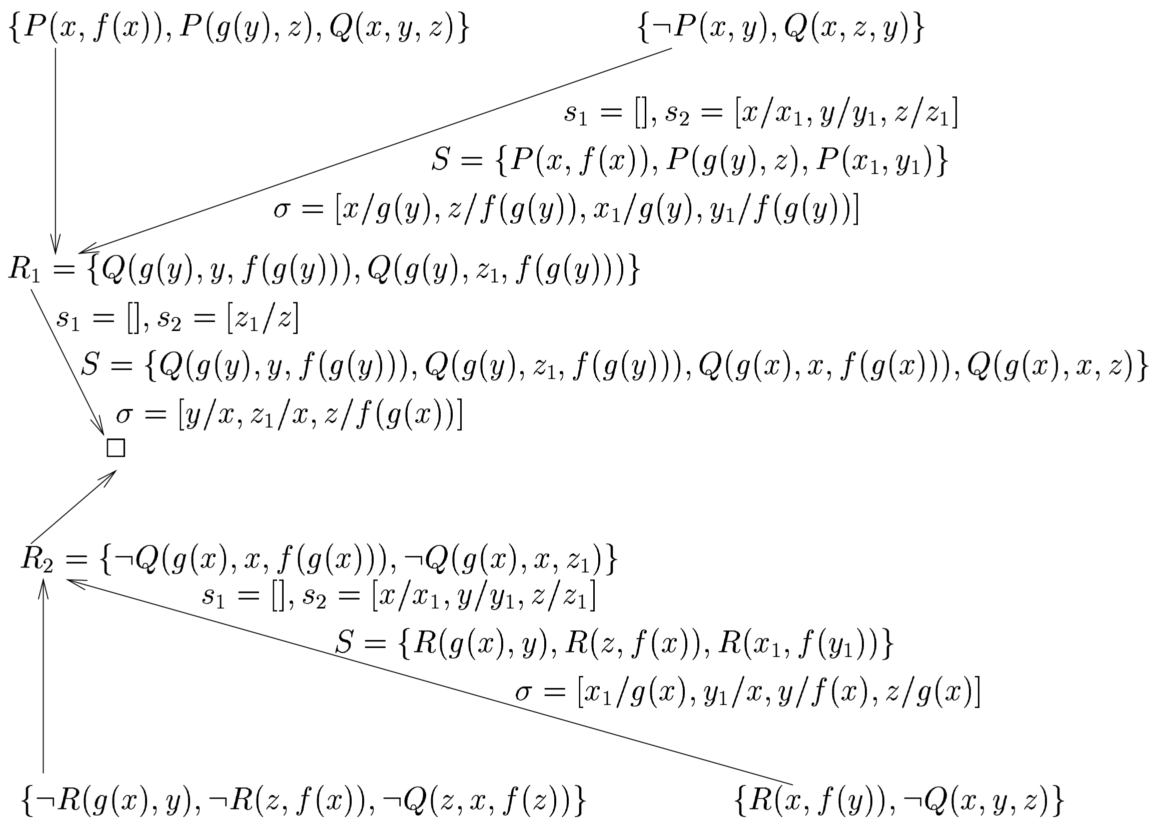

Figure 1: The answer to Question 7

$\{\neg P(x, y), Q(x, z, y)\}, \{\neg R(g(x), y), \neg R(z, f(x)), \neg Q(z, x, f(z))\}, \{R(x, f(y)), \neg Q(x, y, z)\}\}$ is unsatisfiable. For each resolution step specify the relabelings, the unification set and the mgu. Do the minimal number of unifications. Draw your tree on a blank sheet of paper.

Answer: The tree is shown in Figure 1.
Grading Criteria: 1. 5 points for each correct resolution step leading to $\square$.

2. For each step, 0.5 points for relabelings, 0.5 points for S, 1.5 points for mgu and 2.5 points for the resolvent.

3. If both parents of a resolvent are wrong, no credit is given for the step.

4. If one of the parent clauses is wrong you cannot get more than 3 points for the resolvent, even if it is correct.

5. -2 points for each extra resolution step.

Question 8. (16 points)

We create the data base shown below.

```
arc(a,b).
arc(b,b).
arc(b,c).
arc(c,b).
path1(X,X).
path1(X,Y) :- path1(X,Z), arc(Z,Y).
path2(X,X).
path2(X,Y) :- arc(X,Z), path2(Z,Y).
```

What will be printed out by the queries below?

Write yes, no, or out of local stack next to the query.

?- path1(a,c). yes

?- path2(a,c). out of local stack

?- path1(c,a). out of local stack

?- path2(c,a). out of local stack

Grading Criteria: 4 points for each right answer

Question 9. (15 points)

Prove the Compactness Theorem for FOL: A set of formulas F is unsatisfiable, iff it has a finite subset that is unsatisfiable. You may use any theorem in the book except this one. Write your answer on a blank sheet of paper.

Proof: We need to show that a set of FOL formulas S is unsatisfiable, iff it has a finite subset that is unstatisfiable. The if part is easy because whenever a subset of S is unstiafiable, so is S.

For the other part, assume that S is unsatisfiable. Then we Skolemize every formula in the set S. We use a new set of function symbols, g_i^n, $i, n \in N$ to carry out the Skolemization. Now, S is satisfiably equivalent to a set of clauses T.

The Herbrand Theorem is valid for infinite sets of clauses, so the expansion $E[T]$ is unsatisfiable. By the Resolution Theorem for propositional

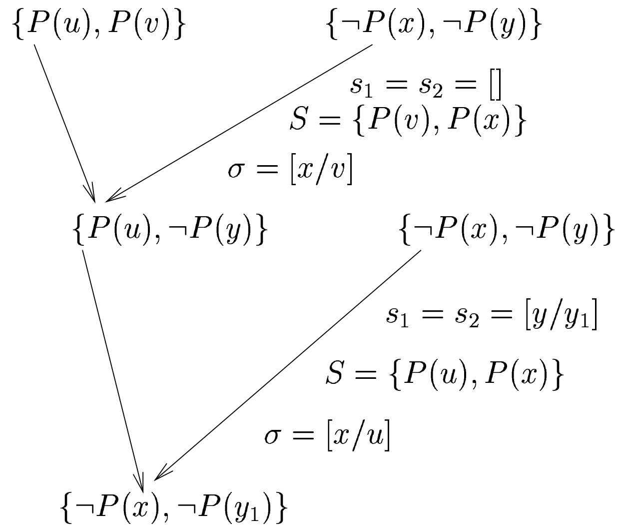

Figure 2: The binary resolution is incomplete

logic, $\Box \in E[T, n]$ for some $n \in N$. Then, we use the Lifting Lemma to raise the derivation tree of $\Box$ from propositional logic to FOL. Let t be that tree and $C_1, \ldots, C_n$ be the leaves of that tree. Let $F_1, \ldots F_n$ be n formulas in S such that C_i is a conjunct of the Skolem form of F_i, $0 \leq i \leq n$. We claim that $U = \{F_1, \ldots, F_n\}$ is unsatisfiable. U is satisfiably equivalent to its set of clauses V. That set is unsatisfiable, since $\Box \in Res^*[V]$. So, U is also unsatisfiable.
Q.E.D.

Question 10 (15 points)

We can define binary resolution for FOL by unifying only one literal from C_1 with one literal from C_2. Prove that the binary resolution is incomplete by exhibiting a set of clauses that yields the box when we apply the full resolution, but does not produce the box when we do the binary resolution.
Solution: Take $S = \{\{P(u), P(v)\}, \{\neg P(x), \neg P(v)\}\}$.

Figure 2 shows that we cannot derive $\Box$ using binary resolution. In all cases, the parent clauses have 2 literals each. We eliminate the 2 literals that we unify and get a resolvent with 2 literals. So, we never get the empty clause.

Figure 3 shows a derivation of $\Box$ from S using full resolution.

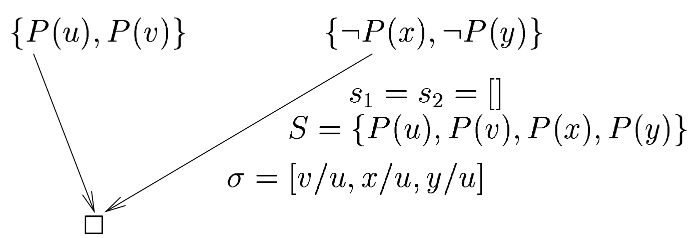

Figure 3: The derivation of $\Box$ from S