COT 3420
SUMMER A 2003
Section 2

EXAM # 1 ANSWERS

Question 1.(20 points)

1\. a 2. b 3. c 4. a 5. c 6. d 7. c 8. d 9. b 10. d

Question 2. (20 points)

The proof is by structural induction on F.

Case 1: F is an atom. Then $F = P_i$ for some $i \in N$. The only suffixes of F are $S = \lambda$ and $S = P_i$. In either case, $n[), S] = n[con, S] = n[(, S] = 0$. So, the inequality holds.

Case 2: $F = \neg G$. The suffixes of F are the suffixes of G and $S = \neg G$.

Subcase 2.1: S is a suffix of G. Then, by induction hypothesis,

$n[), S] \geq n[con, S] \geq n[(, S]$.

Subcase 2.2: $S = \neg G$. Since G is a suffix of G, we have

(1) $n[), G] \geq n[con, G] \geq n[(, G]$

We compute $n[(, S]$, $n[con, S]$, and $n[), S]$. Since $S = \neg G$, and $\neg$ is not one of the symbols $($, $)$, $\vee$, $\wedge$, $\longrightarrow$, $\longleftrightarrow$, we get (2)-(4).

(2) $n[(, S] = n[(, G]$
(3) $n[), S] = n[), G]$
(4) $n[con, S] = n[con, G]$

Now,

$n[), S] = n[), G]$ by (3)
$\geq n[con, G]$ by (1)
$= n[con, S]$ by (4)

So,

(5) $n[), S] \geq n[con, S]$.

At the same time,

$n[con, S] = n[con, G]$ by (4)
$\geq n[(, G]$ by (1)
$= n[(, S]$ by (2)

We get

(6) $n[con, S] \geq n[(, S]$.

From (5) and (6) we have

(7) $n[), S] \geq n[con, S] \geq n[(, S]$.

Cases 3,4,5, 6: $F = (GCH)$. Let S be a suffix of F. We have 4 subcases: $S = \lambda$, $S = J)$ with J a suffix of H, $S = ICH)$ with I a suffix of G, and $S = F$.

Subcase 3.1: $S = \lambda$. This subcase was treated in Case 1.

Subcase 3.2: $S = J)$ with J a suffix of H. We apply the induction hypothesis to the suffix J of H and get (10).

(10) $n[), J] \geq n[con, J] \geq n[(, J]$.

Now we relate the S counts to the J counts. We get

(11) $n[), S] = n[), J] + 1$ because $S = J)$

(12) $n[con, S] = n[con, J]$ because $S = J)$

(13) $n[(, S] = n[(, J]$ because $S = J)$

Now,

$n[), S] = n[), J] + 1$ by (11)

$> n[), J]$

$\geq n[con, J]$ by (10)

$= n[con, S]$ by (12)

So,

(14) $n[), S] > n[con, S]$.

At the same time,

$n[con, S] = n[con, J]$ by (12)

$\geq n[(, J]$ by (10)

$= n[(, S]$ by (13)

We get

(15) $n[con, S] \geq n[con, S]$.

(14) and (15) give the required inequality.

Subcase 3.3: $S = ICH)$ with I a suffix of G. We apply the IH to the suffix H of H and to the suffix I of G. We get (20) and (21).

(20) $n[), H] \geq n[con, H] \geq n[(, H]$

(21) $n[), I] \geq n[con, I] \geq n[(, I]$

Now we compute the counts of $S = ICH)$.

(22) $n[), S] = n[), I] + n[), H] + 1$

(23) $n[con, S] = n[con, I] + 1 + n[con, H]$

(24) $n[(, S] = n[(, I] + n[(, H]$

Now,

$n[), S] = n[), I] + n[), H] + 1$ by (22)

$\geq n[con, I] + n[), H] + 1$ by (21)

$\geq n[con, I] + n[con, H] + 1$ by (20)

$= n[con, S]$ by (23).

So, we got

(25) $n[(, S] \geq n[con, S]$

At the same time,

$n[con, S] = n[con, I] + 1 + n[con, H]$ by (23)

$> n[con, I] + n[con, H]$

$\geq n[(, I] + n[con, H]$ by (21)

$\geq n[(, I] + n[(, H]$ by (20)

$= n[(, S]$ by (24)

The end expression give us

(26) $n[con, S] > n[(, S]$

Subcase 3.4: $S = F$. We apply the IH to the suffix H of H and to the suffix G of G. We get (30) and (31).

(30) $n[), H] \geq n[con, H] \geq n[(, H]$

(31) $n[), G] \geq n[con, G] \geq n[(, G]$

Now we compute the counts of $S = (GCH)$.

(32) $n[), S] = n[), G] + n[), H] + 1$

(33) $n[con, S] = n[con, G] + 1 + n[con, H]$

(34) $n[(, S] = 1 + n[(, G] + n[(, H]$

Now,

$n[), S] = n[), G] + n[), H] + 1$ by (32)

$\geq n[con, G] + n[), H] + 1$ by (31)

$\geq n[con, G] + n[con, H] + 1$ by (30)

$= n[con, S]$ by (33).

So, we got

(35) $n[(, S] \geq n[con, S]$

At the same time,

$n[con, S] = n[con, G] + 1 + n[con, H]$ by (33)

$\geq n[(, G] + 1 + n[con, H]$ by (31)

$\geq n[(, G] + 1 + n[(, H]$ by (30)

$= n[(, S]$ by (34)

The end expression give us

(36) $n[con, S] > n[(, S]$

(35) and (36) give us the double inequality.

Grading Criteria 1. Listing the 6 cases: 2 points.

2. Case 1: 2 points.

3. Case 2: 5 points. Listing the 2 subcases - 1point, Subcase 2.1 - 2 points, Subcase 2.2 - 2points.

4. Cases 3,4,5,6: 11 points. Listing the 4 subcases - 2 points, Subcase 3.1 - 1 point, Subcase 2.2 - 2 points, the other 2 subcases, 3 points each.

5. There is a penalty for ambiguity, bad style, and mistakes of any sort.

Question 3. (10 points)

$F = \neg(((P_0 \wedge \neg((P_9 \longleftrightarrow P_{10}) \longrightarrow P_2)) \longleftrightarrow \neg(P_1 \longleftrightarrow ((\neg P_3 \longrightarrow P_4) \vee P_{11}))) \wedge (P_5 \vee (P_6 \longleftrightarrow (\neg P_7 \longrightarrow P_8))))$.

Grading Criteria: 1 point off for each out of place symbol.

Question 4. (15 points)

Proof by induction.

Basis: $n = 0$. The equation reduces to

$0^3 - 16 * 0$ is divisible by 3,

or

0 is divisible by 3.

This is true because $0 = 3 * 0$.

Inductive Step: Assume that

(IH) $n^3 - 16n$ is divisible by 3.

This means that for some integer k,

(1) $n^3 - 16n = 3k$

Now we compute $(n+1)^3 - 16(n+1)$.

$(n+1)^3 - 16(n+1) = n^3 + 3n^2 + 3n + 1 - 16(n+1)$ by the binomial expansion

$= n^3 + 3n^2 + 3n + 1 - 16n - 16$

$= (n^3 - 16n) + (3n^2 + 3n - 15)$ by grouping

$= 3k + (3n^2 + 3n - 15)$ by (1)

$= 3k + 3(n^2 + n - 5)$ by distributivity

$= 3(k + n^2 + n - 5)$ by distributivity

The equality $(n+1)^3 - 16(n+1) = 3(k + n^2 + n - 5)$ tells us that $(n+1)^3 - 16(n+1)$ is divisible by 3.

Grading Criteria: 1. Listing the 2 cases - 2 points.

2. Basis - 3 points.

3. Inductive Step - 10 points.

Question 5. (10 points)

$S = \{P_1, P_2, P_3, P_0, P_4, (P_1 \vee P_2), \neg P_3, \neg P_1, \neg P_4, ((P_1 \vee P_2) \longrightarrow \neg P_3), (P_1 \wedge P_3), (P_0 \wedge \neg P_4), \neg (P_0 \wedge \neg P_4), (\neg P_1 \vee \neg (P_0 \wedge \neg P_4)), ((P_1 \wedge \neg P_3) \longrightarrow (\neg P_1 \vee \neg (P_0 \wedge \neg P_4))), (((P_1 \vee P_2) \longrightarrow \neg P_3) \longleftrightarrow ((P_1 \wedge \neg P_3) \longrightarrow (\neg P_1 \vee \neg (P_0 \wedge \neg P_4)))), \neg (((P_1 \vee P_2) \longrightarrow \neg P_3) \longleftrightarrow ((P_1 \wedge \neg P_3) \longrightarrow (\neg P_1 \vee \neg (P_0 \wedge \neg P_4))))\}$
Grading Criteria: $\frac{10}{17}$ * (number-of-right-answers - number-of-wrong-answers).