

COT 3420
SUMMER A 2003
Section 1

EXAM # 2 ANSWERS

Question 1.(20 points)

1. c 2. b 3. b 4. a 5. c 6. c 7. a 8. a 9. a 10. a

Grading Criteria: 2 points for each correct answer.

Question 2. (20 points)

Proof: We write $F \diamond G$ for $\neg F \wedge G$. So, the set of connectives is $S = \{\mathbf{T}, \diamond\}$.
The S -formulas are defined below.

1. the atoms are S -formulas,
2. \mathbf{T} is an S -formula,
3. if F and G are S -formulas, so is $F \diamond G$.

We prove, by structural induction, that every formula F has an S -equivalent formula.

Case 1: F is an atom. Then F is an S formula.

Case 2: $F = \neg G$. By induction hypothesis there is an S -formula G_1 such that $G \equiv G_1$. Then

$$\begin{aligned} F &= \neg G \\ &\equiv \neg G_1 && \text{by induction hypothesis} \\ &\equiv \neg G_1 \wedge \mathbf{T} && \text{by the tautology law} \\ &= G_1 \diamond \mathbf{T} && \text{by the definition of } \diamond \end{aligned}$$

The last formula is an S -formula.

Case 3: $F = G \vee H$. By induction hypothesis there are S -formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$$\begin{aligned} F &= G \vee H \\ &\equiv G_1 \vee H_1 && \text{by induction hypothesis} \\ &\equiv \neg \neg (G_1 \vee H_1) && \text{double negation introduction} \\ &\equiv \neg (\neg G_1 \wedge \neg H_1) && \text{De Morgan's Law} \\ &= \neg (G_1 \diamond \neg H_1) && \text{by the definition of } \diamond \\ &\equiv \neg (G_1 \diamond (H_1 \diamond \mathbf{T})) && \text{by Case 2} \\ &\equiv (G_1 \diamond (H_1 \diamond \mathbf{T})) \diamond \mathbf{T} && \text{by Case 2} \end{aligned}$$

The last formula is an S -formula.

Case 4: $F = G \wedge H$. By induction hypothesis there are S -formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$$\begin{aligned}
F &= G \wedge H \\
&\equiv G_1 \wedge H_1 && \text{by induction hypothesis} \\
&\equiv \neg\neg G_1 \wedge H_1 && \text{by double negation introduction} \\
&= \neg G_1 \diamond H_1 && \text{by the definition of } \diamond \\
&\equiv (G_1 \diamond \mathbf{T}) \diamond H_1 && \text{by Case 2}
\end{aligned}$$

The last formula is an S -formula.

Case 5: $F = G \longrightarrow H$. By induction hypothesis there are S -formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$$\begin{aligned}
F &= G \longrightarrow H \\
&\equiv G_1 \longrightarrow H_1 && \text{by induction hypothesis} \\
&\equiv \neg G_1 \vee H_1 && \longrightarrow\text{-elimination} \\
&\equiv \neg G_1 \vee \neg\neg H_1 && \text{double negation introduction} \\
&\equiv \neg(G_1 \wedge \neg H_1) && \text{De Morgan's law} \\
&\equiv \neg(\neg H_1 \wedge G_1) && \text{by the commutativity of } \wedge \\
&= \neg(H_1 \diamond G_1) && \text{by the definition of } \diamond \\
&= (H_1 \diamond G_1) \diamond \mathbf{T} && \text{by Case 2}
\end{aligned}$$

The last formula is S .

Case 6: $F = G \longleftrightarrow H$. By induction hypothesis there are S -formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$$\begin{aligned}
F &= G \longleftrightarrow H \\
&\equiv G_1 \longleftrightarrow H_1 && \text{by induction hypothesis} \\
&\equiv (G_1 \longrightarrow H_1) \wedge (H_1 \longrightarrow G_1) && \text{by } \longleftrightarrow\text{-elim} \\
&\equiv (G_1 \longrightarrow H_1) \wedge ((G_1 \diamond H_1) \diamond \mathbf{T}) && \text{by Case 5} \\
&\equiv (\neg G_1 \vee H_1) \wedge ((G_1 \diamond H_1) \diamond \mathbf{T}) && \longrightarrow\text{-elimination} \\
&\equiv (\neg G_1 \vee \neg\neg H_1) \wedge ((G_1 \diamond H_1) \diamond \mathbf{T}) && \text{double negation introduction} \\
&\equiv \neg(G_1 \wedge \neg H_1) \wedge ((G_1 \diamond H_1) \diamond \mathbf{T}) && \text{De Morgan's law} \\
&\equiv \neg(\neg H_1 \wedge G_1) \wedge ((G_1 \diamond H_1) \diamond \mathbf{T}) && \text{by commutativity} \\
&= \neg(H_1 \diamond G_1) \wedge ((G_1 \diamond H_1) \diamond \mathbf{T}) && \text{by the definition of } \diamond \\
&= (H_1 \diamond G_1) \diamond ((G_1 \diamond H_1) \diamond \mathbf{T}) && \text{by the definition of } \diamond
\end{aligned}$$

The last formula is S .

Grading Criteria: 1. Listing the 6 cases : 3.5 points.

2. Case 1: 1 point.
3. Case 2,4: 2.5 points each.
4. Cases 3,5, 6: 3.5 points each.

For cases 2, 3,4,5 the IH and reasons are worth 0.5 points each. You got points off if the S -formulas are too long.

Question 3. (20 points)

Proof: We assume (1), (2), and (3).

(1) $Sat[F \rightarrow G]$

(2) $Sat[H \rightarrow I]$

(3) F and H have no atoms in common.

We will show that $(F \vee H) \rightarrow (G \vee I)$ is satisfiable. We have 2 cases, according to whether $G \vee I$ is satisfiable or not.

Case 1: $G \vee I$ is satisfiable. Then it has a model, say \mathcal{A} . So,

$$\begin{aligned} & \mathcal{A}[(F \vee H) \rightarrow (G \vee I)] \\ &= \mathcal{A}[F \vee H] \boxed{\rightarrow} \mathcal{A}[G \vee I] \quad \text{interpretation of } \rightarrow \\ &= \mathcal{A}[F \vee H] \boxed{\rightarrow} 1 \quad \mathcal{A} \text{ is a model of } G \vee H \\ &= 1 \quad \text{from the table of } \boxed{\rightarrow} \end{aligned}$$

Since \mathcal{A} is a model for $(F \vee H) \rightarrow (G \vee I)$, the formula is satisfiable.

Case 2: $G \vee I$ is unsatisfiable. Then, both G and I are unsatisfiable. From (1), we get that $F \rightarrow G$ has a model, say \mathcal{A} . From (2) we get that $H \rightarrow I$ has a model, say \mathcal{B} . Now,

$$\begin{aligned} & \mathcal{A}[F \rightarrow G] = \mathcal{A}[F] \boxed{\rightarrow} \mathcal{A}[G] \quad \text{interpretation of } \rightarrow \\ &= \mathcal{A}[F] \boxed{\rightarrow} 0 \quad \text{because } G \text{ is unsatisfiable} \end{aligned}$$

Since $\mathcal{A}[F \rightarrow G] = 1$,

(4) $\mathcal{A}[F] = 0$.

By the same reasoning,

(5) $\mathcal{B}[H] = 0$.

Now let us define the truth assignment \mathcal{C} as

$$\mathcal{C}[P_i] = \begin{cases} \mathcal{A}[P_i] & \text{if } P_i \text{ is in } F \\ \mathcal{B}[P_i] & \text{if } P_i \text{ is not in } F \end{cases}$$

The truth assignments \mathcal{A} and \mathcal{C} agree on F , so

(6) $\mathcal{C}[F] = 0$.

By condition (3), none of the atoms of H are in F , so \mathcal{B} and \mathcal{C} agree on H . Then,

(7) $\mathcal{C}[H] = 0$.

From (6) and (7) we get

(8) $\mathcal{C}[F \vee H] = 0$.

From (8) we get

(9) $\mathcal{C}[(F \vee H) \rightarrow (G \vee I)] = 1$.

So, $(F \vee H) \rightarrow (G \vee I)$ is satisfiable.

In both cases, $(F \vee H) \longrightarrow (G \vee I)$ has a model. So, the formula is satisfiable.

Grading Criteria 1. If you did not right **Proof** or **Disproof** before you starting your work, you cannot get more than 2 points.

1. If you make the wrong choice: 3 points.

2. If you make the correct choice : 6 points. Case 1 is worth 3 points and Case 2, 11 points. In Case 2, the derivations $\mathcal{A}[F] = \mathcal{B}[H] = 0$ are worth 2 points each, and the paste-up 7 points.

Question 4. (20 points)

1. b 2. a 3. b 4. c 5. a 6. a 7. b 8. a 9. a 10. b

Grading Criteria: 2 points for each correct answer.

Question 5. (15 points)

$$\begin{aligned}
 F &= \neg[(A \vee \neg B) \longleftrightarrow \neg(B \wedge C)] \\
 &\equiv \neg[(A \vee \neg B) \longrightarrow \neg(B \wedge C)] \wedge [\neg(B \wedge C) \longrightarrow (A \vee \neg B)] && \longleftrightarrow\text{-} \\
 &\text{elimination} \\
 &\equiv \neg[\neg(A \vee \neg B) \vee \neg(B \wedge C)] \wedge [\neg\neg(B \wedge C) \vee (A \vee \neg B)] && \longrightarrow\text{-elimination} \\
 &\equiv \neg[\neg(A \vee \neg B) \vee \neg(B \wedge C)] \vee \neg[\neg\neg(B \wedge C) \vee (A \vee \neg B)] && \text{De Morgan's} \\
 &\text{law} \\
 &\equiv [\neg\neg(A \vee \neg B) \wedge \neg\neg(B \wedge C)] \vee [\neg\neg\neg(B \wedge C) \wedge \neg(A \vee \neg B)] && \text{De Morgan's} \\
 &\text{law twice} \\
 &\equiv [(A \vee \neg B) \wedge (B \wedge C)] \vee [\neg(B \wedge C) \wedge (\neg A \wedge \neg\neg B)] && \text{double negation} \\
 &\text{elimination, De Morgan's law} \\
 &\equiv [(A \vee \neg B) \wedge B \wedge C] \vee [(\neg B \vee \neg C) \wedge \neg A \wedge B] && \text{De Morgan's law, double} \\
 &\text{negation elimination} \\
 &\equiv (A \vee \neg B \vee \neg B \vee \neg C) \wedge (A \vee \neg B \vee \neg A) \wedge (A \vee \neg B \vee B) \wedge (B \vee \neg B \vee \\
 &\neg C) \wedge (B \vee \neg A) \wedge (B \vee B) \wedge (C \vee \neg B \vee \neg C) \wedge (C \vee \neg A) \wedge (C \vee B) && \text{by} \\
 &\text{generalized distributivity} \\
 &\equiv (A \vee \neg B \vee \neg C) \wedge (\neg A \vee B) \wedge B \wedge (\neg A \vee C) \wedge (B \vee C) && \text{tautology} \\
 &\text{removal, idempotency, commutativity} \\
 &\equiv (A \vee \neg B \vee \neg C) \wedge B \wedge (\neg A \vee C) && \text{absorbtion}
 \end{aligned}$$

Grading: You got credit up the line where you made the first error. For each equivalence you got 1.5 points (up to 9), and 1.5 points for stating the reasons.