COT 3420
SUMMER A 2003
SECTION 1

ANSWERS TO EXAM # 3

Question 1.(20 points)

1\. b 2. c 3. b 4. c 5. a 6. c 7. a 8. d 9. c 10. a

Question 2. (15 points)

We prove, by induction on $n \in N$, that $Tree[S, n] = Res^n[S]$.

Basis: $n = 0$. The S-trees of height 0 are the clauses in S, so $Tree[S, 0] = S$. By definition, $Res^0[S] = S$. So, $Tree[S, 0] = Res^0[S]$.

Inductive Step: Assume that $Tree[S, n] = Res^n[S]$. We will show that

(1) $Tree[S, n + 1] \subseteq Res^{n+1}[S]$

and

(2) $Res^{n+1}[S] \subseteq Tree[S, n + 1]$.

We show (1) first. Let $C \in Tree[S, n + 1]$. Then C has an S-derivation tree of height less than or equal to $n + 1$. Let t be such a tree. If the height of t is less than or equal to n, $C \in Tree[S, n]$. By IH, $C \in Res^n[S]$, so $C \in Res^{n+1}[S]$.

Let us now assume that t has height $n + 1$. Then the tree has the form shown in Figure 1. The heights of t_0 and t_1 are less than or equal to n, so they belong to $Tree[S, n]$. By IH, $C_0, C_1 \in Res^n[S]$. Then $C \in Res[Res^n[S]]$, i.e. $C \in Res^{n+1}[S]$.

In both cases, $C \in Res^{n+1}[S]$. Since C is arbitrary, $Tree[S, n + 1] \subseteq Res^{n+1}[S]$.

Let us show (2). Let C be a clause in $Res^{n+1}[S]$. Then $C \in Res^n[S]$ or C is a resolvent of two clauses from $Res^n[S]$. If $C \in Res^n[S]$, then $C \in Tree[S, n]$ by IH. Since $Tree[S, n] \subseteq Tree[S, n + 1]$, $C \in Tree[S, n + 1]$.

Now let us assume that C is a resolvent of the clauses C_0 and C_1 of $Res^n[S]$. By IH, C_0 and C_1 have resolution trees t_0, respectively t_1, of height less than or equal to n. Then let t be the resolution tree from Figure 1. The height of t is less than or equal to $n + 1$ and it is an S-resolution tree. So, $C \in Tree[S, n + 1]$.

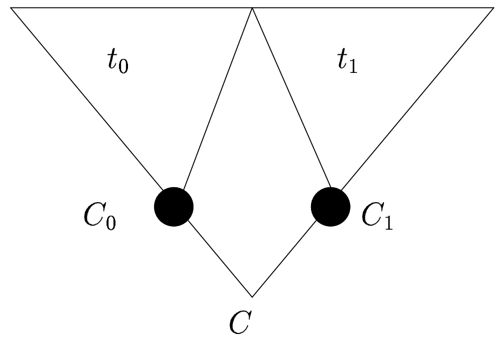

Figure 1: The tree from Question 2

In both cases $C \in Tree[S, n+1]$. Since C is arbitrary, $Res^{n+1}[S] \subseteq Tree[S, n+1]$.

From (1) and (2) we get $Tree[S, n+1] = Res^{n+1}[S]$.

Grading Criteria: 1. Just trying: 2 points.

2. Case $n = 0$: 3 points.
3. Proving (1): 5 points.
4. Proving (2): 5 points.
5. Listing the IH: 2 points.

Question 3. (15 points)

The proof is by structural induction on F.

Case 1: F is an atomic formula. Then, F contains no $\forall$'s and no $\exists$'s, so $F^* = F$. So, F^* is a formula because F is.

Case 2: $F = \neg G$. By IH, G^* is a formula. Since $\neg \neq \forall$, and $\neg \neq \exists$, $F^* = \neg G^*$. Since G^* is a formula, so is $\neg G^*$.

Cases 3,4,5,6: $F = (GCH)$, where C is one of the connectives $\vee, \wedge, \longrightarrow, \longleftrightarrow$. Since (is not a quantifier symbol, C is neither a quantifier symbol nor a variable, and) is not a variable, these symbols will not be deleted. So, $F^* = (G^*CH^*)$. By IH both G^* and H^* are formulas. Then, so is $F^* = (G^*CH^*)$.

Cases 7,8: $F = QxG$, where Q is a quantifier. The deletions do not overlap, so $F^* = G^*$. By IH, G^* is a formula. Then, so is F^*.

Grading Criteria: 1. Listing the cases: 4 points.

2. Case 1: 1 point.
3. Case 2: 3 points.
4. Cases 3,4,5,6: 4 points.

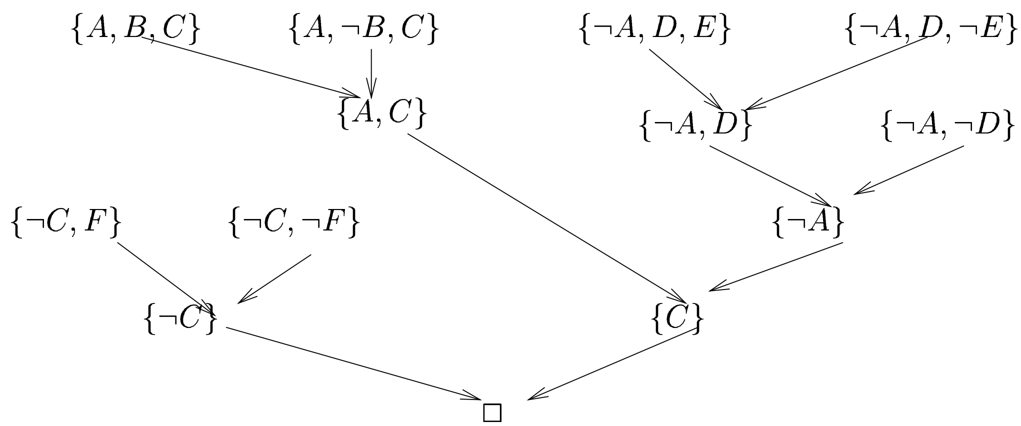

Figure 2: The tree for Question 4

5. Cases 7,8: 3 points.
6. Just trying: 2 points.

Question 4. (20 points)

The resolution tree is shown in Figure 2.

Grading Criteria: 1. 3 points for each correct resolvent. (up to 6)

2. -5 points for each wrong resolvent.

3. 2 points bonus to everyone who tried to solve it.

Question 5. (20 points)

We will show that S is satisfiable iff at least one of $S_{L=0}$, $S_{L=1}$ is satisfiable.
$\Longrightarrow$: Assume that S is satisfiable. Let $\mathcal{A}$ be a model of S. Then $\mathcal{A}[L] = 0$ or $\mathcal{A}[L] = 1$. We will show that $\mathcal{A}[L] = 0$ implies that $S_{L=0}$ is satisfiable. The part that $\mathcal{A}[L] = 1$ implies the satisfiability of $S_{L=1}$ is similar.

Let C be a clause in $S_{L=0}$. If C belongs to S, i.e. it is in the 4th category, then $\mathcal{A}$ is a model of C. If C is not in S, then C is obtained by removing L from a clause in S, i.e. $C \cup \{L\} \in S$. Since $\mathcal{A}$ models $C \cup \{L\}$ and $\mathcal{A}$ does not model L, $\mathcal{A}$ models C.

In both cases, $\models_{\mathcal{A}} C$. Since C is arbitrary in $S_{L=0}$, $\mathcal{A}$ is a model of $S_{L=0}$.
$\Longleftarrow$: Assume that one of $S_{L=0}$, $S_{L=1}$ is satisfiable. The proofs are similar, so let us assume that $S_{L=1}$ is satisfiable. Let $\mathcal{A}$ be a model of $S_{L=1}$ and $\mathcal{B}$ a truth assignment that is just like $\mathcal{A}$, except that $\mathcal{B}[L] = 1$. We will show that $\mathcal{B}$ models S. Let C be a clause in S.

If C belongs to the first 2 categories, then it contains L, so $\mathcal{B}[C] = 1$.

If C belongs to the 3rd category, then $C - \{\bar{L}\}$ is a clause in $S_{L=1}$. Neither L nor $\bar{L}$ occur in $C - \{\bar{L}\}$, so $\mathcal{A}$ and $\mathcal{B}$ agree on $C - \{\bar{L}\}$. Then $\mathcal{B}$ is a model of $C - \{\bar{L}\}$, so it models C.

If C belongs to the 4th category, then it does not contain neither L nor $\bar{L}$. So, $\mathcal{A}$ and $\mathcal{B}$ agree on C. Since C is in $S_{L=1}$, $\mathcal{A}[C] = 1$. Then, $\mathcal{B}[C] = 1$.

Grading Criteria: 1. 2 points for trying.

2. 10 points for each implication.

Question 6. (15 points)

1. $\mathcal{A}[f(x)] = 5$
2. $\mathcal{A}[f(g(a, y))] = 3$
3. $\mathcal{A}[P(x, y)] = 0$
4. $\mathcal{A}[\forall x P(y, x)] = 1$
5. $\mathcal{A}[\exists y P(x, y)] = 0$

Grading Criteria: 3 points for each right answer.