COT 3420
SUMMER A 2003
SECTION 2

ANSWERS TO EXAM # 3

Question 1.(20 points)

1. a 2. a 3. b 4. a 5. a 6. c 7. b 8. d 9. c 10. d

Grading Criteria: 2 points for each correct answer.

Question 2. (15 points)

Proof: Since $Res[Res^*[S]] = Res^*[S] \cup \{R | R$ is a resolvent of two clauses C_1 and C_2 of $Res^*[S]\}$,

(1) $Res^*[S] \subseteq Res[Res^*[S]]$.

We need to show that

(2) $Res[Res^*[Res^*[S]] \subseteq Res^*[S]$.

Let $C \in Res[Res^*[S]]$. Then either $C \in Res^*[S]$ or C is a resolvent of two clauses in $Res^*[S]$.

Case 1: $C \in Res^*[S]$. Then we have no problem.

Case 2: C is a resolvent of two clauses, C_0 and C_1 of $Res^*[S]$. Since C_0, C_1 are in $Res^*[S] = \bigcup_{i=0}^{\infty} Res^i[S]$, there are indices j, k such that $C_0 \in Res^j[S]$ and $C_1 \in Res^k[S]$. Let l be the largest of j, k. Then, $Res^j[S] \subseteq Res^l[S]$ and $Res^k[S] \subseteq Res^l[S]$. So, both C_0 and C_1 are in $Res^l[S]$. Then, their resolvent C is in $Res[Res^l[S]] = Res^{l+1}[S]$. Since $Res^{l+1}[S] \subseteq Res^*[S]$, $C \in Res^*[S]$.

In both cases $C \in Res^*[S]$. Since C is arbitrary, we have (2).

Grading Criteria: 1. Just trying: 2 points.

2. Showing that $Res^*[S] \subseteq Res[Res^*[S]]$: 3 points.

3. Proving (2): 12 points.

Question 3. (20 points)

The proof is by structural induction on t.

Case 1: t is a variable. Then t is a symbol that is neither a parenthesis nor a function symbol. Its prefixes are $S = \lambda$ and $S = t$. In either case, $n[fun, S] = n[(, S] = n[), S] = 0$.

Case 2: t is a function constant. Then t is a symbol that is neither a parenthesis nor a function symbol of arity greater than 0. So, $n[fun, t] = n[(, t] = n[), t] = 0$. Then, for any substring S of t, in particular for prefixes, $n[fun, S] = n[(, S] = n[), S] = 0$.
Case 3: $t = f(t_1, \ldots, t_n)$ where f is a function symbol of arity greater than 0 and $t_1, \ldots, t_n$ are terms. Let S be a prefix of t. Then S belongs to one of the 4 subcases below.
Subcase 3.1: $S = \lambda$. Then $n[fun, S] = n[(, S] = n[), S] = 0$.
Subcase 3.2: $S = f$. Then $n[fun, S] = 1$ and $n[(, S] = n[), S] = 0$.
Subcase 3.3: $S = f(t_1, \ldots, T_i$ where T_i is a prefix of t_i. This subcase covers all prefixes that end on positions 1, 2, ...,$length[t]-2$ of the t. If $S = f($ then $i = 1$ and $T_1 = \lambda$. If S ends with a proper prefix of t_i, then $S = f(t_1, \ldots, T_i$ where T_i is the proper prefix of t_i. If S ends on the comma separating t_i and t_{i+1}, then $S = f(t_1, \ldots, t_i, T_{i+1}$ where $T_{i+1} = \lambda$. If S contains all of t except the final closed parenthesis, then $S = f(t_1, \ldots, T_n$ where $T_n = t_n$.

Now let us show that $n[fun, S] \geq n[(, S]$.

$n[fun, S] = n[fun, f(t_1, \ldots, T_i]$

$= 1 + n[fun, t_1] + \ldots + n[fun, T_i]$ 1 is for f

$\geq 1 + n[(, t_1] + \ldots + n[(, T_i]$ IH applied to the prefixes t_1 of t_1,...,T_i of t_i

$= n[(, f(t_1, \ldots, T_i]$ 1 is for the leftmost (

$= n[(, S]$

Now let us prove that $n[(, S] \geq n[), S]$.

$n[(, S] = n[(, f(t_1, \ldots, T_i]$

$= 1 + n[(, t_1] + \ldots + n[(, T_i]$ 1 for the leftmost (

$> n[(, t_1] + \ldots + n[(, T_i]$ remove the 1

$\geq n[), t_1] + \ldots + n[), T_i]$ by IH applied to the prefixes t_1 of t_1, ..., T_i of t_i

$= n[), f(t_1, \ldots, T_i]$

$= n[), S]$.

Subcase 3.4: $S = t$. The proof is similar to Subcase 3.3.

Grading Criteria: 1. Listing the 3 cases: 2 points.

2. Cases 1,2: 2 points each.
3. Subcases 3.1, 3.2: 2 points each.
4. Subcases 3.3, 3.4: 4 points each.
5. Listing the 4 subcases of Case 3: 2 points.
6. Just trying: 2 points.

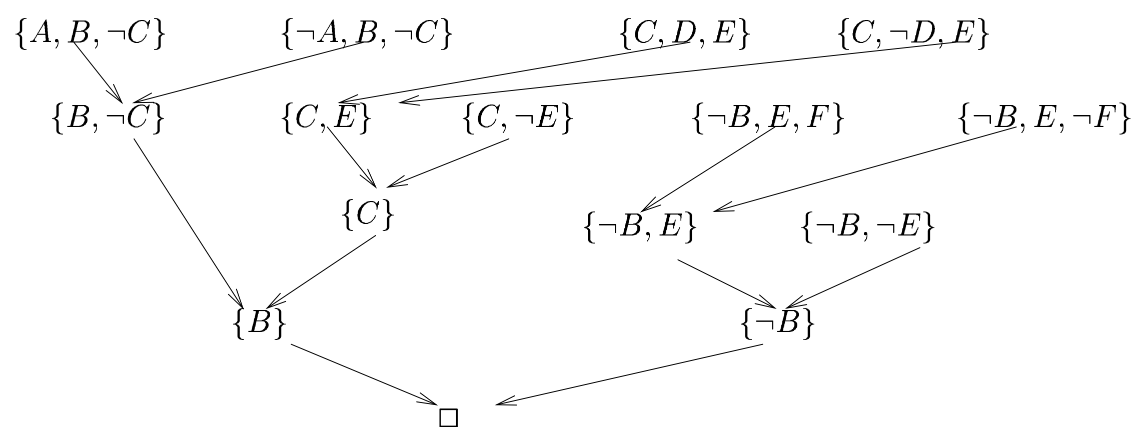

Figure 1: The tree for Question 4

Question 4. (20 points)

The resolution tree is shown in Figure 1.

Grading Criteria: 1. 2.5 points for each correct resolvent. (up to 7)

2. -4 points for each wrong resolvent.

3. 2.5 points bonus to everyone who tried to solve it.

Question 5. (15 points)

By The Compactness Theorem, S is satisfiable iff every finite subset of S is satisfiable.

We will show that

(1) if S is satisfiable, then $\bigwedge_{i=0}^{n} F_i$ is satisfiable for every n, and

(2) if $\bigwedge_{i=0}^{n} F_i$ is satisfiable for infinitely many n's than S is satisfiable.

We show (1) first. So, assume that S is satisfiable. By The Compactness Theorem, every finite subset of S is satisfiable. In particular, the subsets $\{F_0, \ldots, F_n\}$ are satisfiable. Since $\{F_0, \ldots, F_n\} \equiv \bigwedge_{i=0}^{n} F_i$, $\bigwedge_{i=0}^{n} F_i$ is satisfiable. So, every $\bigwedge_{i=0}^{n} F_i$ is satisfiable.

Now let us show (2). Assume that infinitely many conjunctions $\bigwedge_{i=0}^{n} F_i$ are satisfiable. Now let T be a finite subset of S. If T is empty we have no problem because every empty subset is satisfiable. If T is not empty, it contains a formula F_k such that k is the largest index of all formulas of T. Since $\bigwedge_{i=0}^{n} F_i$ is satisfiable for infinitely many n's, we choose an n larger than k. Then $T \subseteq \{F_0, \ldots, F_n\}$ because the indices of the T-formulas are less than or equal to k. Since $\bigwedge_{i=0}^{n} F_i$ is satisfiable, so is $\{F_0, \ldots, F_n\}$. Then its subset T is also satisfiable.

Since T is an arbitrary finite subset of S, every finite subset of S is satisfiable. By The Compactness Theorem, S is satisfiable.

Grading Criteria: 1. 2 points for trying.

2. 5 points for mentioning The Compactness Theorem.

3. 2 points for observing that $\bigwedge_{i=0}^{n} F_i \equiv \{F_0, \ldots, F_n\}$.

Question 6. (15 points)

1. $\mathcal{A}[f(x)] = 5$
2. $\mathcal{A}[f(g(a, y))] = 5$
3. $\mathcal{A}[P(x, y)] = 1$
4. $\mathcal{A}[\forall x P(y, x)] = 1$
5. $\mathcal{A}[\exists y P(x, y)] = 0$

Grading Criteria: 3 points for each right answer.