COT 3420
SUMMER A 2003

ANSWERS TO PRACTICE EXAM # 4

Question 1.(5 points)
Skolemize the formula $F = \exists x \forall y \exists z \exists u \forall v \exists w F^M[x, y, z, u, v, w]$.
$F^* = \forall y \forall v F^M[a, y, f(y), g(y), v, h(y, v)]$.

Question 2. (10 points)
Rectify the formula $F = \forall x(P(x, y) \vee \exists x P(x, z)) \wedge \forall y(\neg P(y, x) \vee \forall z \neg P(z, f(y)))$.
$F^* = \forall u(P(u, y) \vee \exists v P(v, z)) \wedge \forall w(\neg P(w, x) \vee \forall z_1 \neg P(z_1, f(w)))$.

Question 3. (5 points)
Close the formula $F = \forall z \exists v F^M[x, y, z, u, v, w]$.
$F^* = \exists x \exists y \exists u \exists w \forall z \exists v F^M[x, y, z, u, v, w]$.

Question 4. (10 points)
Prove that if x is not free in G, then
$\forall x F \longrightarrow G \equiv \exists x(F \longrightarrow G)$.

A Syntactic Proof:
$\forall x F \longrightarrow G$
$\equiv \neg \forall x F \vee G$ $\longrightarrow$-elimination
$\equiv \exists x \neg F \vee G$ $\neg \forall x F \equiv \exists x \neg F$
$\equiv \exists x(\neg F \vee G)$ x is not free in G
$\equiv \exists x(F \longrightarrow G)$ $\longrightarrow$-introduction

A semantic Proof:
Let $\mathcal{A}$ be a structure with universe D.
$\mathcal{A}[\forall x F \longrightarrow G] = 1$
iff $\mathcal{A}[\forall x F] = 0$ or $\mathcal{A}[G] = 1$ interpretation of $\longrightarrow$
iff there exists $d \in D$ such that $\mathcal{A}_{[x \leftarrow d]}[F] = 0$, or $\mathcal{A}[G] = 1$ interpretation of $\forall x$
iff there exists $d \in D$ such that, $\mathcal{A}_{[x \leftarrow d]}[F] = 0$ or $\mathcal{A}[G] = 1$
iff there exists $d \in D$ such that, $\mathcal{A}_{[x \leftarrow d]}[F] = 0$ or $\mathcal{A}_{[x \leftarrow d]}[G] = 1$ x is not free in G, so $\mathcal{A}$ and $\mathcal{A}_{[x \leftarrow d]}$ agree on G

iff there exists $d \in D$ such that $\mathcal{A}_{[x \leftarrow d]}[F \longrightarrow G] = 1$ interpretation of $\longrightarrow$

iff $\mathcal{A}[\exists x(F \longrightarrow G)] = 1$ interpretation of $\exists x$

Question 5. (15 points)

Let S be the set that contains all atomic formulas, the empty clause, and the operators $\longrightarrow$ and $\exists x$, where x can be any variable. Show, by structural induction, that S is adequate.

Proof: We show by structural induction that every formula F has an equivalent S-formula.

Case 1: F is atomic. Then F is an S-formula.

Case 2: $F = \neg G$. By IH there is an S-formula G_1 such that $G \equiv G_1$. Then

$F = \neg G$

$\equiv \neg G_1$ by IH

$\equiv \neg G_1 \vee \Box$ contradiction law

$\equiv G_1 \longrightarrow \Box$ $\longrightarrow$-introduction

The last formula is S.

Case 3: $F = G \vee H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$. Then

$F = G \vee H$

$\equiv G_1 \vee H_1$ by IH

$\equiv \neg\neg G_1 \vee H_1$ $\neg\neg$ introduction

$\equiv \neg G_1 \longrightarrow H_1$ $\longrightarrow$-introduction

$\equiv (G_1 \longrightarrow \Box) \longrightarrow H_1$ Case 2

The last formula is an S-formula.

Case 4: $F = G \wedge H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$F = G \wedge H$

$\equiv G_1 \wedge H_1$ by IH

$\equiv \neg\neg(G_1 \wedge H_1)$ $\neg\neg$ introduction

$\equiv \neg(\neg G_1 \vee \neg H_1)$ De Morgan's law

$\equiv \neg(G_1 \longrightarrow \neg H_1)$ $\longrightarrow$- introduction

$\equiv (G_1 \longrightarrow \neg H_1) \longrightarrow \Box$ Case 2

$\equiv (G_1 \longrightarrow (H_1 \longrightarrow \Box) \longrightarrow \Box$ Case 2

Case 5: $F = G \longrightarrow H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$. The

$F = G \longrightarrow H$

$\equiv G_1 \longrightarrow H_1$.

The last formula is an S-formula.

Case 6: $F = G \longleftrightarrow H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$. Then

$F = G \longleftrightarrow H$

$\equiv G_1 \longrightarrow H_1$ by IH

$\equiv (G_1 \longrightarrow H_1) \wedge (H_1 \longrightarrow G_1)$ $\longleftrightarrow$-elim

Then apply Case 4 to get rid of $\wedge$.

Case 7: $F = \forall x G$. By IH there is an S-formula G_1 such that $G \equiv G_1$. Then

$F = \forall x G$

$\equiv \forall x G_1$

$\equiv \neg\neg\forall x G_1$ $\neg\neg$-introduction

$\equiv \neg\exists x \neg G_1$

$\equiv \exists x \neg G_1 \longrightarrow \Box$ Case 2

$\equiv \exists x (G_1 \longrightarrow \Box) \longrightarrow \Box$ Case 2

Case 8: $F = \exists x G$. By IH there is an S-formula G_1 such that $G \equiv G_1$. Then $F \equiv \exists x G_1$.

Question 6. (15 points)

Construct a derivation tree of $\Box$ from $S = \{\{\neg P(x,y), \neg P(y,z), Q(x, f(z))\}, \{P(x,y), Q(x,z)\}, \{\neg Q(a,x), \neg R(x,y)\}, \{R(f(x),a), R(y,x)\}\}$.

Use the minimal number of steps.

The tree is shown in Figure 1.

Question 7. (10 points)

Write $E(F,2)$ for $F = \forall x \forall y((P(a, f(x)) \vee P(f(y), x)) \wedge \neg P(b,y))$.

$D(F,2) = \{a, b, f(a), f(b), f^2(a), f^2(b)\}$. There are 36 formulas in $E(F,2)$.

$E(F,2) = \{F^M[a,a], F^M[a,b], F^M[a,f(a)], F^M[a,f(b)], F^M[a,f^2(a)], F^M[a,f^2(b)],$
$F^M[b,a], F^M[b,b], F^M[b,f(a)], F^M[b,f(b)], F^M[b,f^2(a)], F^M[b,f^2(b)],$
$F^M[f(a),a], F^M[f(a),b], F^M[f(a),f(a)], F^M[f(a),f(b)], F^M[f(a),f^2(a)], F^M[f(a),f^2(b)],$
$F^M[f(b),a], F^M[f(b),b], F^M[f(b),f(a)], F^M[f(b),f(b)], F^M[f(b),f^2(a)], F^M[f(b),f^2(b)],$
$F^M[f^2(a),a], F^M[f^2(a),b], F^M[f^2(a),f(a)], F^M[f^2(a),f(b)], F^M[f^2(a),f^2(a)],$
$F^M[f^2(a),f^2(b)], F^M[f^2(b),a], F^M[f^2(b),b], F^M[f^2(b),f(a)], F^M[f^2(b),f(b)],$
$F^M[F^2(b),f^2(a)], F^M[f^2(b),f^2(b)]\}$.

Bonus Question (15 points)

Let C be a clause and s a substitution. We call the clause $s[C]$ a factoring of C. For example, $\{\neg P(x,x)\}$ is a factoring of $\{\neg P(x,y), \neg P(y,z)\}$

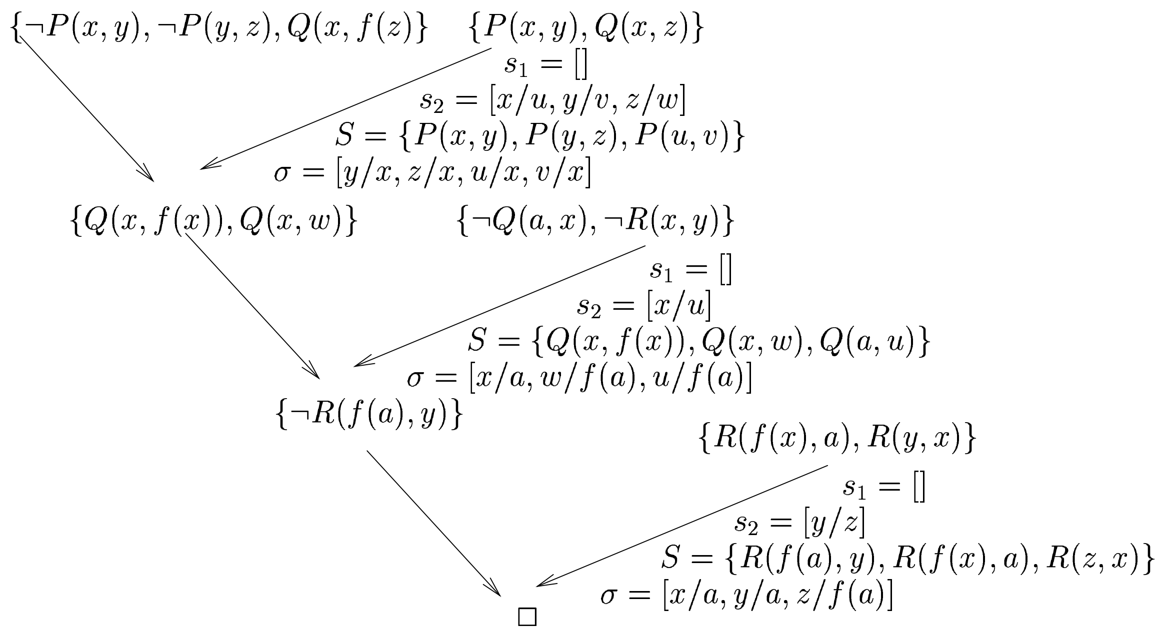

Figure 1: The answer to question 4

because $\{\neg P(x,y), \neg P(y,z)\}[y/x, z/x] = \{\neg P(x,x)\}$. We recall that the binary resolution unifies one literal from clause C_1 with one literal from clause C_2. Prove that the full resolution can be implemented with binary resolution and factoring.

Proof: Let us assume that C_1 and C_2 are two clauses with no variables in common. Let $S = \{P_1, \ldots, P_n, Q_1, \ldots, Q_m\}$ be the set of atoms, with $S_1 = \{P_1, \ldots P_n\}$ a subset of C_1 and the complement of $S_2 = \{Q_1, \ldots, Q_m\}$ a subset of C_2. Let σ be an mgu of S produced by the book algorithm. Since S is unifiable, so are its subsets S_1 and S_2. Let s_1 be an mgu of S_1 and s_2 an mgu of S_2 given by the book algorithm. Then $s_1[S_1]$ and $s_2[S_2]$ have no variables in common!

By the property of the mgu's

(1) $\sigma = \pi \circ s_1$

and

(2) $\sigma = \rho \circ s_2$.

Since s_1 and s_2 have no variables in common, $\sigma = ((\pi \uparrow Var[s_1]) \cup (\rho \uparrow Var[s_2])) \circ (s_1 \cup s_2)$. Let $\alpha = (\pi \uparrow Var[s_1]) \cup (\rho \uparrow Var[s_2])$. Since σ is a unifier of S, $\alpha(s_1 \cup s_2)[S_1] = \alpha(s_1 \cup s_2)[S_2]$. Let β be an mgu of $s_1[S_1]$ and

$s_2[S_2]$. Since β is the mgu, $\alpha = \gamma \circ \beta$ for some γ. Then

(3) $\sigma = \gamma \circ \beta \circ (S_1 \cup s_2)$.

This means that the resolvent R of C_1 and C_2 on σ can be implemented by factoring S_1 on s_1, factoring S_2 on s_1 and unifying $s_1[C_1]$ and $s_2[C_2]$ on the binary $\{s_1[S_1], \neg s_2[S_2]\}$.