COT 3420
Section U1
Spring 2005

EXAM # 1 ANSWERS

Question 1. (36 points)

1. b 2. d 3. b 4. a 5. c 6. a 7. a 8. c 9. d 10. a 11. d 12. b

Grading Criteria: 3 points for each correct answer, 0 for wrong choice, no choice, or multiple choices.

Question 2. (40 points)

Prove that for all formulas F, $n[atom, F] = n[(, F] + 1$. Recall that $n[atom, F]$ is the number of atom occurrences in F.

Proof by structural induction on F.

Case 1: F is an atom, say P_i. Since P_i is a symbol different from (, $n[(, F] = 0$. At the same time P_i is an atom, so $n[atom, F] = 1$. So,

$n[atom, F] = n[(, F] + 1$

becomes $1 = 0 + 1$, which is true.

Case 2: $F = \neg G$. By induction hypothesis on G,

(IH) $n[atom, G] = n[(, G] + 1$

We relate the F counts to the G counts.

(1) $n[(, F] = n[(, G]$ because $\neg \neq ($

(2) $n[atom, F] = n[atom, G]$ because $\neg$ is not an atom

Now,

$n[atom, F] = n[atom, G]$ by(2)

$= n[(, G] + 1$ by (IH)

$= n[(, F] + 1$ by (1)

So, $n[atom, F] = n[(, F] + 1$.

Cases 3,4,5,6: $F = (GCH)$ for $C \in \{\vee, \wedge, \longrightarrow, \longleftrightarrow\}$. By induction hypotheses,

(IH on G) $n[atom, G] = n[(, G] + 1$

and

(IH on H) $n[atom, H] = n[(, H] + 1$

Now we relate the F counts to the G and H counts.

(3) $n[(, F] = 1 + n[(, G] + n[(, H]$ because $F = (GCH)$, $C \neq ($, and $) \neq ($

(4) $n[atom, F] = n[atom, G] + n[atom, H]$ because $F = (GCH)$, and $(, C,)$, are not atoms

Now we compute $n[atom, F]$.

$n[atom, F] = n[atom, G] + n[atom, H]$ by (4)

$= [n[(, G] + 1] + [n[(, H] + 1]$ by IH on G and H

$= [1 + n[(, G] + n[(, H]] + 1$ by grouping

$= n[(, F] + 1$ by (3)

So, $n[atom, F] = n[(, F] + 1$. **Q.E.D.**

Grading Criteria:

1. Listing the cases: 3 points.
2. Case 1: 4 points
3. Case 2: 14 points
   - the IH : 3 points
   - the relations (1) and (2): 6 points
   - the derivation of $n[atom, F] = n[(, F] + 1$: 3 points
   - the reasons for the equalities: 2 points
4. Cases 3, 4,5,6: 19 points
   - the IH's : 6 points
   - the relations (3) and (4): 6 points
   - the derivation of $n[atom, F] = n[(, F] + 1$: 4 points
   - the reasons for the equalities: 3 points

Question 3. (15 points)

Draw the formula tree of $F = (((\neg P_0 \vee P_1) \wedge \neg(P_2 \longleftrightarrow \neg P_3)) \vee ((P_2 \longleftrightarrow P_3) \wedge (P_4 \vee \neg(P_1 \longrightarrow (P_5 \vee (P_6 \longleftrightarrow \neg P_3))))))$.

The tree is found in Figure 1.

Grading Criteria : - 2 points for each wrong label or arrow.

Question 4. (15 points)

Write the formula that corresponds to the tree from Figure 2.

$\neg(((\neg P_0 \vee ((P_4 \vee P_1) \wedge P_3)) \wedge \neg(P_2 \wedge P_6)) \longleftrightarrow (P_4 \longleftrightarrow ((\neg P_3 \longrightarrow P_7) \wedge (P_5 \vee P_4))))$.

Grading Criteria: - 2 points for each symbol out of place.

Question 5. (15 points)

Prove the following proposition:

If S is a non-empty suffix of a formula F and $n[(, S] = n[), S]$, then S is formula.

Proof

Since this problem is worth only 15 points you don't need to include all details. We do the proof by structural induction.

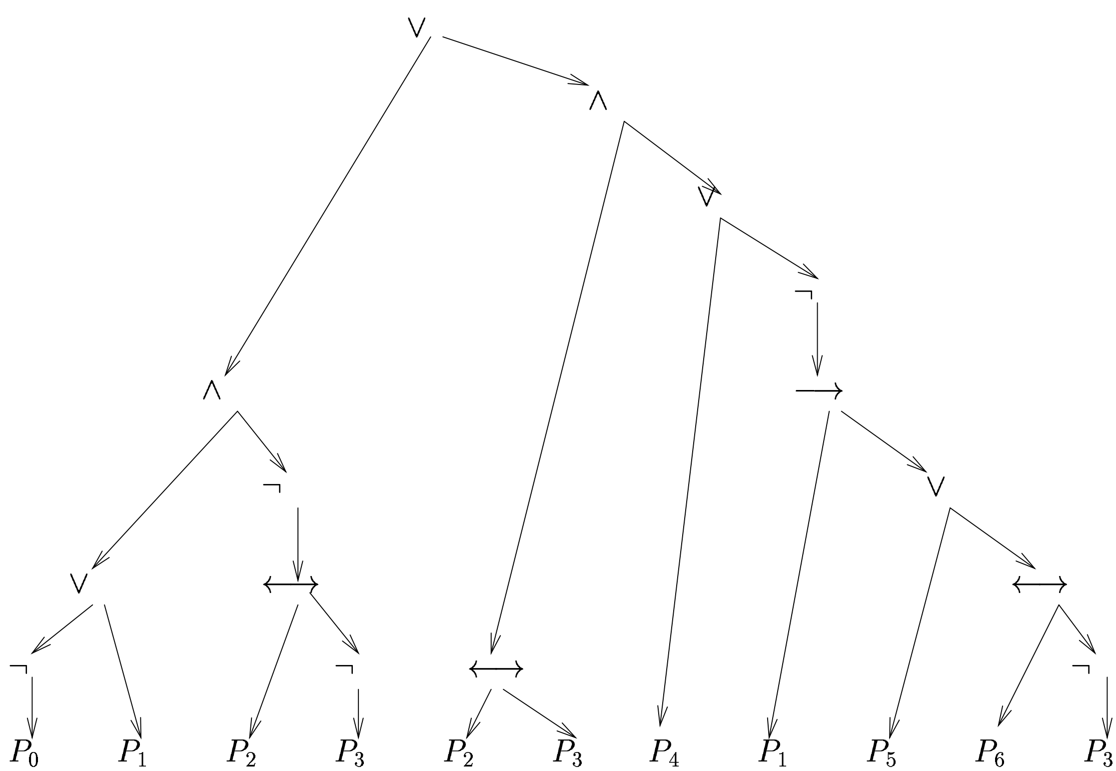

Figure 1: The answer to Question 3

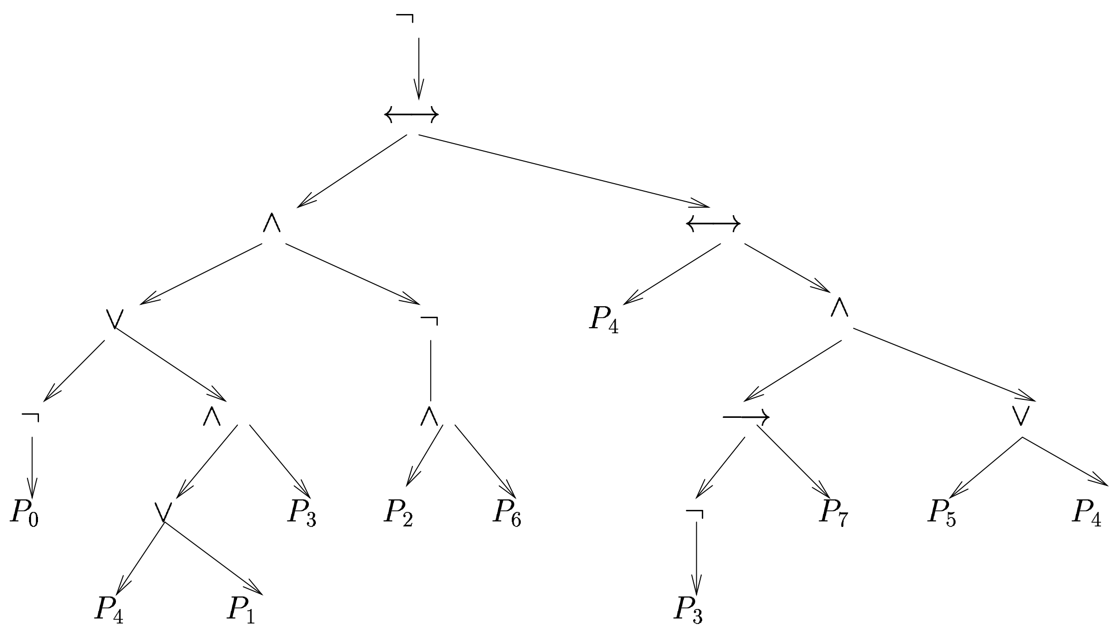

Figure 2: The tree for Question 4

Case 1: F is an atom. Since atoms are symbols, the only non-empty suffix of F is F itself. Since F is a formula, we are done. Notice that we did not need the condition $n[(,S] = n[),S]$ for this case.
Case 2: $F = \neg G$. The non-empty suffixes of F are the non-empty suffixes of G and F itself. If S is a non-empty suffix of G with $n[(,S] = n[),S]$, then S is a formula by the induction hypothesis (the proposition is true for G). If $S = F$ then S is formula because F is a formula.
Cases 3,4,5,6: $F = (GCH)$ where C is a binary connective. We will show the only non-empty suffix of F that has an equal number of left and right parentheses is F.

The non-empty suffix of F are

3.1: $S = U)$ with U a suffix of H. By Lemma 1.2.8, U has at least as many)'s as ('s. So, $n[),S] = n[),U)]$

$= n[),U] + 1$

$> n[),U]$

$\geq n[(,U]$ Lemma 1.2.8

$= n[(,U)]$ $) \neq ($

$= n[),S]$

So, S has more right parentheses than left parentheses and the proposition is vacuously true.

3.2: $S = UCH)$ with U a suffix of G. We apply Lemma 1.2.8 to the suffixes U of G and H and H and get that both have at least as many right parentheses as left parentheses. A proof similar to 3.1 shows that S has more)'s than ('s, so the proposition is true.

3.3: $S = F$. Then S is a formula, so we have nothing to prove.
Grading Criteria: Case 1, 3 points; Case 2, 4 points, Case 3, 8 points.