COT 3420
Section U1
Spring 2005

FINAL EXAM ANSWERS

Question 1. (20 points)

Let $F = \forall x \forall y F^M$ be a formula with matrix F^M, and f,g be two function symbols of arity 2. Let u and v be two variables that do not occur in F.

Prove that $F \models \forall u \forall v F^M[x/f(u,v), y/g(u,v)]$.

Proof: Let $\mathcal{A}$ be a model of F. Then, from the interpretation of $\forall x$, $\forall y$ we get that (1) holds for all $d, e \in |\mathcal{A}|$.

(1) $\mathcal{A}_{[x\leftarrow d][y\leftarrow e]}[F^M] = 1$

Now let $a, b \in |\mathcal{A}|$.

$\mathcal{A}_{[u\leftarrow a][v\leftarrow b]}[F^M[x/f(u,v), y/g(u,v)]$

$= \mathcal{A}_{[u\leftarrow a][v\leftarrow b]}[F^M[y/g(u,v)][x/f(u,v)]$ because x is not in $g(u,v)$

$= \mathcal{A}_{[u\leftarrow a][v\leftarrow b][x\leftarrow \mathcal{A}_{[u\leftarrow a][v\leftarrow b]}[f(u,v)]]}[F^M[y/g(u,v)]]$ by The Translation Lemma, since $f(u,v)$ is free for x in $F^M[y/g(u,v)]$

$= \mathcal{A}_{[u\leftarrow a][v\leftarrow b][x\leftarrow f^{\mathcal{A}}[a,b]]}[F^M[y/g(u,v)]]$ from the interpretation of $f(u,v)$

$= \mathcal{A}_{[u\leftarrow a][v\leftarrow b][x\leftarrow f^{\mathcal{A}}[a,b]][y\leftarrow \mathcal{A}_{[u\leftarrow a][v\leftarrow b][x\leftarrow f^{\mathcal{A}}[a,b]]}[g(u,v)]]}[F^M]$ $g(u,v)$ is free for y in F^M

$= \mathcal{A}_{[u\leftarrow a][v\leftarrow b][x\leftarrow f^{\mathcal{A}}[a,b]][y\leftarrow g^{\mathcal{A}}[a,b]]}[F^M]$ we evaluate $g(u,v)$

$= \mathcal{A}_{[x\leftarrow f^{\mathcal{A}}[a,b]][y\leftarrow g^{\mathcal{A}}[a,b]]}[F^M]$ because u and v do not occur in F^M

So, we got

(2) $\mathcal{A}_{[u\leftarrow a][v\leftarrow b]}[F^M[x/f(u,v), y/g(u,v)] = \mathcal{A}_{[x\leftarrow f^{\mathcal{A}}[a,b]][y\leftarrow g^{\mathcal{A}}[a,b]]}[F^M]$

We use (1) with $d = f^{\mathcal{A}}[a,b]$ and $e = g^{\mathcal{A}}[a,b]$ and get (3)

(3) $\mathcal{A}_{[x\leftarrow f^{\mathcal{A}}[a,b]][y\leftarrow g^{\mathcal{A}}[a,b]]}[F^M] = 1$

Since a and b are arbitrary, (3) is true for all $a, b \in |\mathcal{A}|$, i.e. $\mathcal{A}$ is a model of $\forall u \forall v F^M[x/f(u,v), y/g(u,v)]$. **Q.E.D.**

Grading Criteria

The problem here is sloppiness.

1. Listing (1) is worth 2 points
2. Each of the 6 steps is worth 2 points.
3. the wrap up is worth 3 points.
4. The arguments to the steps is worth 3 points.
5. Just trying 1-4 points.

Question 2. (10 points)

Rectify the formula $F = \forall x\{\forall y[\exists x(P(x,y) \vee \neg Q(z)) \wedge \forall z(\neg P(x,z) \vee Q(y))] \wedge \exists z[\forall x(P(x,z) \vee Q(y)) \wedge \exists z(\neg Q(z) \vee \neg P(z,x))]\}$.

We relabel the first $\forall x$ by $\forall x_1$, the $\exists x$ by $\exists x_2$, $\forall z$ by $\forall z_1$, the first $\exists z$ by $\exists z_2$, the second $\exists z$ by $\exists z_3$ and $\forall y$ by $\forall y_1$.

$F \equiv \forall x_1\{\forall y_1[\exists x_2(P(x_2,y_1)\vee\neg Q(z))\wedge\forall z_1(\neg P(x_1,z_1)\vee Q(y_1))]\wedge\exists z_2[\forall x(P(x,z_2)\vee Q(y)) \wedge \exists z_3(\neg Q(z_3) \vee \neg P(z_3,x_1))]\}$

Grading Criteria: -1 point for each wrong symbol, i.e. symbol that was not changed, or was changed incorrectly. This include the arguments of the quantifiers.

Question 3. (5 points)

Close the formula $F = \forall x\exists y F^M[x,y,z,u,v,w]$ where F^M, the matrix of F, has free occurrences of x,y,z,u,v,w.

The closure is $\exists z\exists u\exists v\exists w F$.

Grading Criteria

1. Quantifying a variable that is already quantified in F: -1 point for each occurrence.

2. Changing the quantifier of a variable that is quantified in F: -1 point for each occurrence.

3. Inserting $\exists z, \exists u, \exists v, \exists w$ after one of the existing quantifiers, $\forall x$ or $\exists y$: -1 point for each insertion.

4. Using $\forall$ instead of $\exists$: -1 point for each substitution.

Question 4. (10 points)

Skolemize the formula $F = \exists x\exists y\forall z\forall u\exists v\exists w F^M$, where F^M, the matrix of F, contains the constants a and b and the function symbols f and h.

The Skolemized formula is $\forall z\forall u F^M[x/c, y/d, v/g(z,u), w/i(z,u)]$

Grading Criteria

1. 2.5 points for each successful replacement.

2. -2 points for each illegal replacement like using symbols that are already in F^M, overloading a function symbol, or using functions of wrong arity.

3. If 2 errors occured in the substitution of a variable, then there is no credit for that substitution. For example $v/f(z)$ brings no credit because f already occurs in F^M and v must be replaced by a function of arity 2.

Question 5. (10 points)

x	$f^{\mathcal{A}}[x]$
3	3
4	5
5	4

$f^{\mathcal{A}}$

$x \backslash y$	3	4	5
3	3	4	5
4	4	5	3
5	5	3	4

$g^{\mathcal{A}}$

$x \backslash y$	3	4	5
3	1	0	0
4	0	0	1
5	0	1	0

$P^{\mathcal{A}}$

x	$Q^{\mathcal{A}}[x]$
3	0
4	0
5	1

$Q^{\mathcal{A}}$

Figure 1: Tables for Question 7

Find a prenex form for
$F = \neg\{\forall x[\exists y(P(x,y) \vee \neg Q(x,y)) \wedge \forall z Q(x,z)] \vee \neg\exists u \forall v \exists w(\neg P(u,v) \wedge Q(v,w))\}$.

A prenex form is $\exists z \exists x \exists u \forall y \forall v \exists w F^M$.

Grading Criteria

1. -1 point for each wrong quantifier symbol.
2. -1.5 points for each pair of quantifiers that is out of order; $\exists x$ must precede $\forall y$ and $\exists u$ must precede $\forall v$.
3. -2 points for quantifying a variable twice or for missing a quantifier.
4. -2 points for extra symbols (negations, etc).

Question 6. (10 points)

Find a mgu for $S = \{P(x, f(x,y)), P(g(y), f(g(z),y)), P(u, f(u, h(v)))\}$.

An mgu is $[x/g(h(v)), z/h(v), y/h(v), u/g(h(v))]$.

Grading Criteria

1. If you wrote not unifiable, you get 0 points.
2. Otherwise you get 2.5 points for each variable substitution.
3. -3 points if you left the mgu as a composition of elementary substitutions.
4. -2 points for each repetition *variable/term* in the mgu.

Question 7. (30 points)

The universe of $\mathcal{A}$ is $\{3,4,5\}$ and the $\mathcal{A}$ interpretations of a,x, and y are $a^{\mathcal{A}} = 4$, $x^{\mathcal{A}} = 3$ and $y^{\mathcal{A}} = 5$. The tables for the functions $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ and the predicates $P^{\mathcal{A}}$ and $Q^{\mathcal{A}}$ are displayed in Figure 1.

Evaluate the terms and the formulas below. Do not show your work, just write the answer to the right of the equal sign.

1. $\mathcal{A}[f(f(y))] = 5$
2. $\mathcal{A}[g(g(a,x),y)] = 3$
3. $\mathcal{A}[\neg P(a,x)] = 1$
4. $\mathcal{A}[\forall x P(x, f(x))] = 1$
5. $\mathcal{A}[\exists x \forall y P(x,y)] = 0$
6. $\mathcal{A}[\forall x (Q(x) \vee Q(f(x)))] = 0$

Grading Criteria

1. 5 points for each correct answer.

Question 8. (20 points)

Prove by first order resolution that $\{\{P(x,y), P(y,z), Q(x,z)\}, \{\neg P(x,y), Q(a,x)\}, \{R(x,f(x)), \neg Q(x,a)\}, \{\neg R(a,y), \neg R(x,y), \neg Q(x,x)\}\}$ is unsatisfiable. For each resolution step specify the relabelings, the unification set and the mgu. Do the minimal number of unifications. Draw your tree on a blank sheet of paper.

The answer is diplayed in Figure 2.

Grading Criteria

1. 6.5 points for each correct resolution step (up to 3).
2. The distribution of points for each resolution step:
   1 point for relabelings
   0.5 points for S
   2.5 points for σ
   2.5 points for the resolvent.
3. If one of the parent clauses is missing or is incorrect, that resolution step is ignored.
4. -2 points for every step where more than 2 literals can be unified.

Question 9. (20 points)

We create the data base shown below.

```
arc(a,b).
arc(b,c).
arc(c,a).
path1(X,X).
path1(X,Y) :- arc(X,Z), path1(Z,Y).
path2(X,X).
path2(X,Y):- path2(X,Z), arc(Z,Y).
```

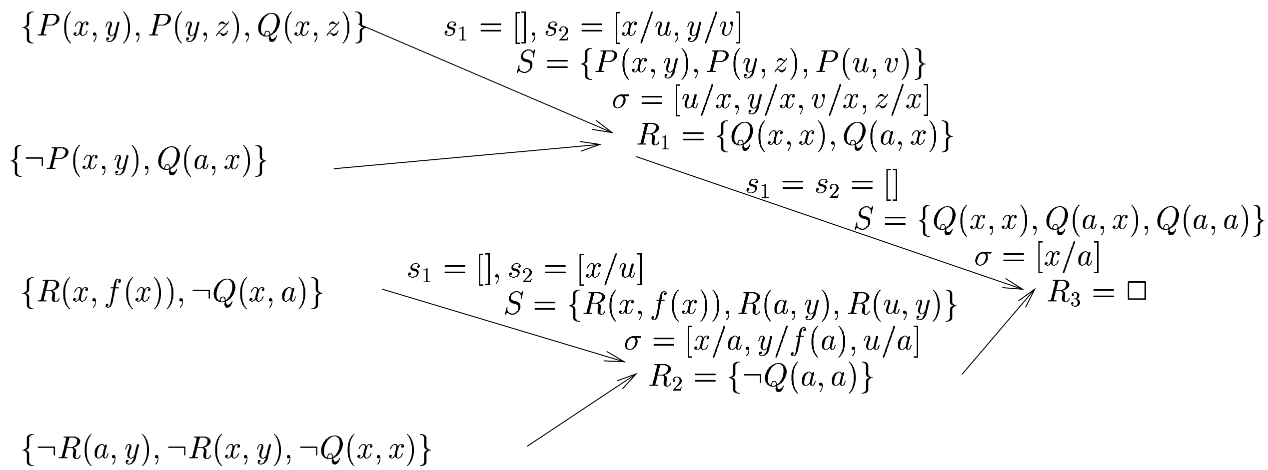

Figure 2: The resolution Tree

What will be printed out by the queries below? Write yes, no, or infinite loop next to the query.

?- path1(a,c). Yes

?- path2(a,c). Yes

?- path1(d,a). No

?- path2(d,a). Infinite loop

Grading Criteria

1. 5 points for each correct answer.

Question 10. (20 points)

The matrix of the formula F is the string obtained from F by deleting all quantifiers Qx. Prove, by structural induction on F, that the matrix of F is a formula. Write your proof on a blank sheet of paper.

Proof: Let $R[F]$ be a matrix of F.

Case 1: F is an atomic formula. Then F has no occurrences of Qx, so $R(F) = F$.

Case 2: $F = \neg G$. Since $\neg$ is not part of a string Qx, $R[\neg G] = \neg R[G]$.

By induction hypothesis, $R[G]$ is formula. Form the definition of a formula, so is $\neg R[G]$.

Cases 3,4,5,6: $F = (GCH)$, where C is one of the binary connectives $\vee, \wedge, \longrightarrow, \longleftrightarrow$. Since neither the parentheses nor C are part of a reduction,

$R[(GCH)] = (R[G]CR[H])$

By IH, both $R[G]$ and $R[H]$ are formulas. Then, so is $(R[G]CR[H])$ by the definition of a formula.

Cases 7, 8: $F = QyG$. Then we delete Qy, and $R[F] = R[G]$. By induction hypothesis, $R[G]$ is a formula.

Grading Criteria:

1. Listing the cases: 3 points.
2. Case 1: 3 points.
3. Case 2: 4 points
4. Cases 3,4,5,6: 6 points.
5. Cases 7,8: 4 points.
6. Stating that the reduced form of $\neg G$ is G, or any other error of this kind: -1.5 points.
7. Not mentioning the induction hypothesis: -1.5 points per occurrence.

Question 11. (20 points)

Prove the consequence below.

$\forall x(F \vee G), \neg F[x/a] \models \exists x G.$

Proof: We need to show that any structure $\mathcal{A}$ that satisfies (1) and (2) satisfies (3).

(1) $\mathcal{A}[\forall x(F \vee G)] = 1$

(2) $\mathcal{A}[\neg F[x/a]] = 1$

(3) $\mathcal{A}[\exists x G] = 1$

The relation (4) is derived from (1) by spelling out the interpretation of $\forall x$.

(4) for all $d \in |\mathcal{A}|$, $\mathcal{A}_{x \leftarrow d]}[(F \vee G)] = 1$

From (4) we get the formula (5) by taking $d = a^{\mathcal{A}}$.

(5) $\mathcal{A}_{[x \leftarrow a^{\mathcal{A}}]}[(F \vee G)] = 1$

From (2) and the interpretation of $\neg$, we get (6).

(6) $\mathcal{A}[F[x/a]] = 0$

Since a is free for x in F, we apply The Translation Lemma and get (7).

(7) $\mathcal{A}[F[x/a]] = \mathcal{A}_{[x \leftarrow a^{\mathcal{A}}]}[F]$.

From (6) and (7) we obtain (8).

(8) $\mathcal{A}_{[x \leftarrow a^{\mathcal{A}}]}[F] = 0$

From (5),(8) and the interpretation of $\vee$ we get (9).

(9) $\mathcal{A}_{[x \leftarrow a^{\mathcal{A}}]}[G] = 1$

Finally, we use the interpretation of $\exists x$ and (9) to get (3).

Grading Criteria

1. If you did not list (1) and (2) you cannot get more than 3 points.
2. If you did not get to (5) you cannot get more than 6 points.
3. Th crux of the proof is (5). It is worth 7 points.

Question 12. (20 points)

Assume that y is not free in G. Prove that $\forall y F \longleftrightarrow G \equiv \forall y(F \longleftrightarrow G)$ is not always true.

Proof: Let us give a counter-example. Let $F = P(y)$ and $G = Q$, a predicate constant. Let $\mathcal{A}$ be a structure with universe $|\mathcal{A}| = \{3, 4\}$, $Q^{\mathcal{A}} = 0$, and $P^{\mathcal{A}}[3] = 0, P^{\mathcal{A}}[4] = 1$.

Then,

$\mathcal{A}[\forall y F] = P^{\mathcal{A}}[3]\boxed{\wedge}P^{\mathcal{A}}[4]$

$= 0\boxed{\wedge}1 = 0. \qquad (1)$

So,

$\mathcal{A}[\forall y F \longleftrightarrow G] = \mathcal{A}[\forall y F]\boxed{\longleftrightarrow}Q^{\mathcal{A}}$

$= 0\boxed{\longleftrightarrow}0 = 1 \qquad (2)$

At the same time,

$\mathcal{A}[\forall y(P(y) \longleftrightarrow Q)]$

$= \mathcal{A}_{[}y \leftarrow 3][P(y) \longleftrightarrow Q]\boxed{\wedge}\mathcal{A}_{[}y \leftarrow 4][P(y) \longleftrightarrow Q]$

$= [P^{\mathcal{A}}[3]\boxed{\longleftrightarrow}Q^{\mathcal{A}}]\boxed{\wedge}[P^{\mathcal{A}}[4]\boxed{\longleftrightarrow}Q^{\mathcal{A}}]$

$= [0\boxed{\longleftrightarrow}0]\boxed{\wedge}[1\boxed{\longleftrightarrow}0]$

$= 1\boxed{\wedge}0 = 0 \qquad (3).$

So, $\mathcal{A}$ is a model of the LHS of $\forall y F \longleftrightarrow G \equiv \forall y(F \longleftrightarrow G)$ and a counter-model of the RHS. **Q.E.D.**

Grading Criteria

1. 10 points for the counter-example, 10 points for the proof.
2. If no attempt at counter-example is made, you get at most 4 points.