COT 3420
Section U01
Fall, 2010

EXAM # 1 ANSWERS

Question 1. (20 points)
1. a 2. b 3. a 4. a 5. c 6. c 7. a 8. b 9. d
10. a

Grading Criteria: 2 points for each correct answer.

Question 2. (20 points)

Prove by structural induction that $|F| = 4*n[con, F]+n[\neg, F]+1$, where $|F|$ is the length (number of symbols) of F, $n[con, F]$ is the number of binary connectives, and $n[\neg, F]$ is the number of negations of F. Do the proof from scratch, without using any of the lemmas and the problems from the book or from the practice exams.

Proof

Case 1: F is an atom. Then $|F| = 1$ and $n[con, F] = n[\neg, F] = 0$. The equation becomes

$1 = 4 * 0 + 0 + 1$

which is true.

Case 2: $F = \neg G$. We have (1) by IH on G.

(1) $|G| = 4 * n[con, G] + n[\neg, G] + 1$

We express the length and the counts of F in function of the corresponding measures of G.

(2) $|F| = |G| + 1$

(3) $n[con, F] = n[con, G]$

(4) $n[\neg, F] = n[\neg, G] + 1$

We use (2), (3), and (4) to replace $|G|, n[con, G], n[\neg, G]$ in (1). We get (5).

(5) $|F| - 1 = 4 * n[con, F] + n[\neg, F] - 1 + 1$

By adding 1 to both sides we have the required equality.

Cases 3, 4, 5, 6: $F = (GCH)$ where C is a binary connective.

By IH we get (6) and (7).

(6) $|G| = 4 * n[con, G] + n[\neg, G] + 1$

(7) $|H| = 4 * n[con, H] + n[\neg, H] + 1$

Now we express the length and the counts of F in function of the lengths and counts of G and H.

(8) $|F| = |G| + |H| + 3$

(9) $n[con, F] = n[con, G] + n[con, H] + 1$

(10) $n[\neg, F] = n[\neg, G] + n[\neg, H]$

Now let us compute $|F|$.

$|F| = |G| + |H| + 3$ by (8)

$= 4 * n[con, G] + n[\neg, G] + 1 + 4 * n[con, H] + n[\neg, H] + 1 + 3$ by (6) and (7)

$= 4 * (n[con, G] + n[con, H] + 1) + (n[\neg, G] + n[\neg, H]) + 1$ by grouping

$= 4 * n[con, F] + n[\neg, F] + 1$ by (9) and (10).

The end expressions of this sequence of equalities is the required identity.

Grading Criteria: 1.Listing the cases: 2 points

2. Case 1: 2 points
3. Case 2: 7 points. The IH is worth 2 points.
4. Case 3: 9 points. The IHs are worth 3 points, the derivation 4.
5. -5 points for writing $F = GCH$.

Question 3. (20 points)

1. b 2. d 3. c 4. c 5. a 6. b 7. b 8. d 9. a 10. b

Grading Criteria: 2 points for each correct answer.

Question 4. (20 points)

Proof: We know (1), (2), and (3).

(1) $\models (F \longrightarrow G)$

(2) F and G have no atoms in common

(3) Sat$[F]$

We need to show (4).

(4) $\models G$

We will show that every truth assignment is a model of G. From (3) we get that F has a model. Let $\mathcal{A}$ be a model.

(5) $\models_{\mathcal{A}} F$

Now let $\mathcal{B}$ be a truth assignment. We use cut and paste to define the truth assignment $\mathcal{C}$.

(6)

$$\mathcal{C}[P_i] = \begin{cases} \mathcal{A}[P_i] & \text{if } P_i \text{ is in } F \\ \mathcal{B}[P_i] & \text{if } P_i \text{ is not in } F \end{cases}$$

From (6) we get that $\mathcal{C}$ and $\mathcal{A}$ agree on F, so we get (7) by the agreement theorem.

(7) $\mathcal{C}[F] = \mathcal{A}[F]$

Since F and G have no atoms in common, $\mathcal{C}$ and $\mathcal{B}$ agree on G. We have (8) by the agreement theorem.

(8) $\mathcal{C}[G] = \mathcal{B}[G]$

We obtain (9) from (1).

(9) $\mathcal{C}[F \longrightarrow G] = 1$

Now, let us compute $\mathcal{C}[F \longrightarrow G] = 1$.

$\mathcal{C}[F \longrightarrow G] = \mathcal{C}[F]\boxed{\longrightarrow}\mathcal{C}[G]$ interpretation of $\longleftrightarrow$

$= \mathcal{A}[F]\boxed{\longrightarrow}\mathcal{C}[G]$ by (7)

$= \mathcal{A}[F]\boxed{\longrightarrow}\mathcal{B}[G]$ by (8)

$= 1\boxed{\longrightarrow}\mathcal{B}[G]$ by (5)

$= \mathcal{B}[G]$ from the table of $\boxed{\longrightarrow}$

So, we get (10).

(10)$\mathcal{C}[F \longrightarrow G] = \mathcal{B}[G]$

From (9) and (10) we have (11).

(11) $\mathcal{B}[G] = 1$

Since $\mathcal{B}$ is arbitrary, we conclude $\models G$. **Q.E.D.**

Grading Criteria:

1. Writing disproof: 2 points and this is all.
2. Writing proof: 5 points + the points for the proof
3. Writing the assumptions: 2 points
4. The cut and paste definition of $\mathcal{C}$: 7 points
5. The rest of the proof: 6 points.

Question 5. (20 points)

$F = \neg\{[(A \vee \neg B) \longrightarrow (C \wedge D)] \wedge [(A \vee C) \longleftrightarrow (B \vee \neg D)]\}$

line 1: $\equiv \neg\{[(A\vee\neg B) \longrightarrow (C\wedge D)]\wedge[(A\vee C) \longrightarrow (B\vee\neg D)]\wedge[(B\vee\neg D) \longrightarrow (A \vee C)]\}$ eliminate $\longleftrightarrow$

line 2: $\equiv \neg\{[\neg(A \vee \neg B) \vee (C \wedge D)] \wedge [\neg(A \vee C) \vee (B \vee \neg D)] \wedge [\neg(B \vee \neg D) \vee (A \vee C)]\}$ eliminate $\longrightarrow$

line 3: $\equiv \neg[\neg(A \vee \neg B) \vee (C \wedge D)] \vee \neg[\neg(A \vee C) \vee (B \vee \neg D)] \vee \neg[\neg(B \vee \neg D) \vee (A \vee C)]\}$ generalized DeMorgan's

line 4: $\equiv [\neg\neg(A \vee \neg B) \wedge \neg(C \wedge D)] \vee [\neg\neg(A \vee C) \wedge \neg(B \vee \neg D)] \vee [\neg\neg(B \vee \neg D) \wedge \neg(A \vee C)]\}$ DeMorgan's 3 times

line 5: $\equiv [(A \vee \neg B) \wedge (\neg C \vee \neg D)] \vee [(A \vee C) \wedge (\neg B \wedge \neg\neg D)] \vee [(B \vee \neg D) \wedge (\neg A \wedge \neg C)]\}$ $\neg\neg$ elim, DeMorgan's, $\neg\neg$ elim, DeMorgan's, $\neg\neg$ elim, DeMorgan's

line 6: $\equiv [(A \vee \neg B) \wedge (\neg C \vee \neg D)] \vee [(A \vee C) \wedge \neg B \wedge D] \vee [(B \vee \neg D) \wedge \neg A \wedge \neg C]\}$ $\neg\neg$ elim

line 7:$\equiv$ $[(A \vee \neg B \vee A \vee C) \wedge (A \vee \neg B \vee \neg B) \wedge (A \vee \neg B \vee D) \wedge (\neg C \vee \neg D \vee A \vee C) \wedge (\neg C \vee \neg D \vee \neg B) \wedge (\neg C \vee \neg D \vee D)] \vee [(B \vee \neg D) \wedge \neg A \wedge \neg C]$ distributivity

line 8: $\equiv [(A \vee \neg B \vee C) \wedge (A \vee \neg B) \wedge (A \vee \neg B \vee D) \wedge (\neg B \vee \neg C \vee \neg D)] \vee [(B \vee \neg D) \wedge \neg A \wedge \neg C]$ idempotency, idempotency, tautology, ordering, tautology

line 9: $\equiv [(A \vee \neg B) \wedge (\neg B \vee \neg C \vee \neg D)] \vee [(B \vee \neg D) \wedge \neg A \wedge \neg C]$ absorbtion

line 10: $\equiv (A \vee \neg B \vee B \vee \neg D) \wedge (A \vee \neg B \vee \neg A) \wedge (A \vee \neg B \vee \neg C) \wedge (\neg B \vee \neg C \vee \neg D \vee B \vee \neg D) \wedge (\neg B \vee \neg C \vee \neg D \vee \neg A) \wedge (\neg B \vee \neg C \vee \neg D \vee \neg C)$ distributivity

line 11: $\equiv (A \vee \neg B \vee \neg C) \wedge (\neg A \vee \neg B \vee \neg C \vee \neg D) \wedge (\neg B \vee \neg C \vee \neg D)$ tautology three times, ordering, idempotency

line 12: $\equiv (A \vee \neg B \vee \neg C) \wedge (\neg B \vee \neg C \vee \neg D)$ absorbtion

Grading Criteria: You get credit to the first line where you made an error.

If the error occured in line 1: 0 points, line 2: 1 point, line 3: 3 points, line 4: 5 points, line 5: 7 points, line 6: 9 points, line 7: 10 points, line 8:

12 points, line 9: 14 points, line 10: 16 points, line 11: 18 points, line 12: 19 points.

If you did not write the reasons for the equivalences, you loose 2 points.