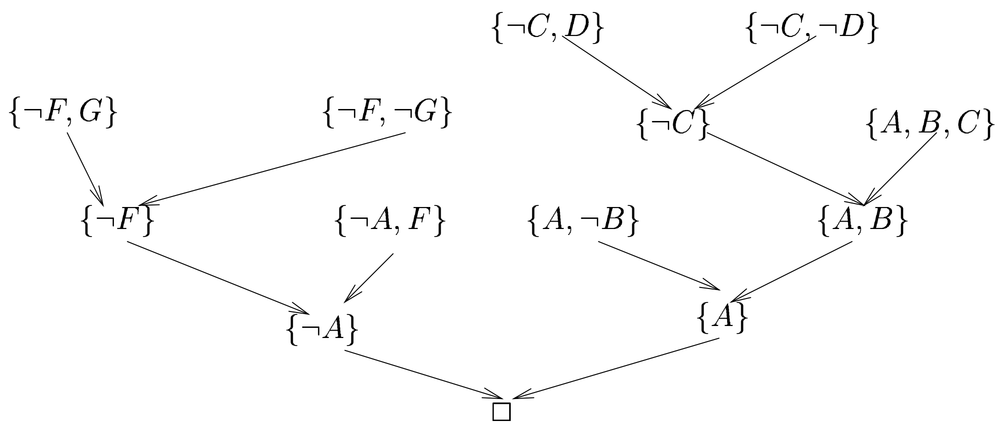

Figure 1: The tree for for Question 2

COT 3420
Section U1
Summer 2005

ANSWERS TO EXAM # 2

Question 1. (36 points)

1. c 2. b 3. a 4. b 5. b 6. c 7. a 8. b 9. d 10. a 11. a 12. e

Grading Criteria: 3 points for each correct answer.

Question 2. (20 points)

The tree is shown in Figure 1.

Grading: 3 points for each correct resolution step (up to 6).

You get 2 extra points if you found the box in 6 steps.

Question 3. (25 points)

Prove, by structural induction, that the connective $F \longrightarrow \neg G$ is adequate.

Proof: Let $S = \{I \longrightarrow \neg J\}$. We will show that for every F there is an S-formula equivalent to it.

Case 1: F is an atomic formula. Then F is an S-formula.

Case 2: $F = \neg G$. By IH there is an S-formula G_1 equivalent to G. Then

$F = \neg G$

$\equiv \neg G_1$ by IH

$\equiv \neg G_1 \vee \neg G_1$ by idempotency

$\equiv G_1 \longrightarrow G_1$ $\longrightarrow$ introduction

The last formula is an S-formula.

Case 3: $F = G \vee H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$F = G \vee H$

$\equiv G_1 \vee H_1$ by IH

$\equiv \neg\neg G_1 \vee \neg\neg H_1$ double negation intro

$\equiv \neg G_1 \longrightarrow \neg\neg H_1$ $\longrightarrow$ intro

$\equiv (G_1 \longrightarrow \neg G_1) \longrightarrow \neg(H_1 \longrightarrow \neg H_1)$ Case 2

The last formula is an S-formula.

Case 4: $F = G \wedge H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$F = G \wedge H$

$\equiv G_1 \wedge H_1$ by IH

$\equiv \neg\neg(G_1 \wedge H_1)$ double negation introduction

$\equiv \neg(\neg G_1 \vee \neg H_1)$ De Morgan's law

$\equiv \neg(G_1 \longrightarrow \neg H_1)$ $\longrightarrow$ intro

$\equiv (G_1 \longrightarrow \neg H_1) \longrightarrow \neg(G_1 \longrightarrow \neg H_1)$ Case 2

The last formula is an S-formula.

Case 5: $F = G \longrightarrow H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$F = G \longrightarrow H$

$\equiv G_1 \longrightarrow H_1$ by IH

$\equiv G_1 \longrightarrow \neg\neg H_1$ double negation introduction

$\equiv G_1 \longrightarrow \neg(H_1 \longrightarrow \neg H_1)$ Case 2

The last formula is an S-formula.

Case 6: $F = G \longleftrightarrow H$. By IH there are S-formulas G_1 and H_1 such that $G \equiv G_1$ and $H \equiv H_1$.

$F = G \longleftrightarrow H$

$\equiv G_1 \longleftrightarrow H_1$ by IH

$\equiv (G_1 \longleftrightarrow H_1) \wedge (H_1 \longleftrightarrow G_1)$ $\longleftrightarrow$ elimination

We know how to eliminate $\wedge$ by Case 4 and $\neg$ by Case 2. So, we can get an S-formula equivalent to F.

Grading Criteria:

1. Listing the cases: 2 points
2. Case 1: 2 points
3. Case 2: 7 points.

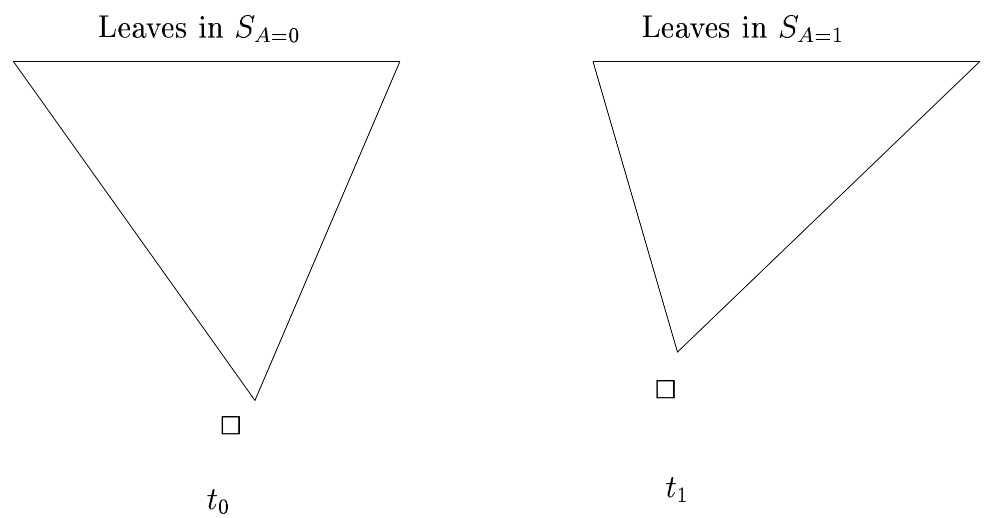

Figure 2: The trees t_0 and t_1 from Question 4

3.1. The IH: 1 points.
3.2. The derivation: 4 points
3.3. The reasons for the derivations: 2 points
4. Case 3 (or Case 4) : 9 points
3.1. The IH: 2 points.
3.2. The derivation: 5 points
3.3. The reasons for the derivations: 2 points
5. You can reduce the other 3 cases to the 2 above: 5 points

Question 4. (25 points)

Let S be a set of clauses with n atoms. Prove that S is unsatisfiable iff $\Box \in Res^n[S]$.

Proof: If $\Box \in Res^n[S]$, then $\Box \in Res^*[S]$, since $Res^n[S] \subseteq Res^*[S]$. Then, The Resolution Theorem says that S is unsatisfiable.

Now, let us assume that S is unsatisfiable. We prove the proposition by induction on n.

Basis: $n = 0$. Then S is unsatisfiable iff $\Box \in S$.

Inductive Step: Assume that the proposition is true for all sets T with n or fewer atoms. Let S be an unsatisfiable set with $n+1$ atoms. Let A be an atom in S, and let us construct the sets $S_{A=0}$ and $S_{A=1}$. Since S is unsatisfiable, so are $S_{A=0}$ and $S_{A=1}$. Since the last 2 sets have at most n atoms, we apply the IH to them. We get the trees t_0 and t_1 shown in Figure 2.

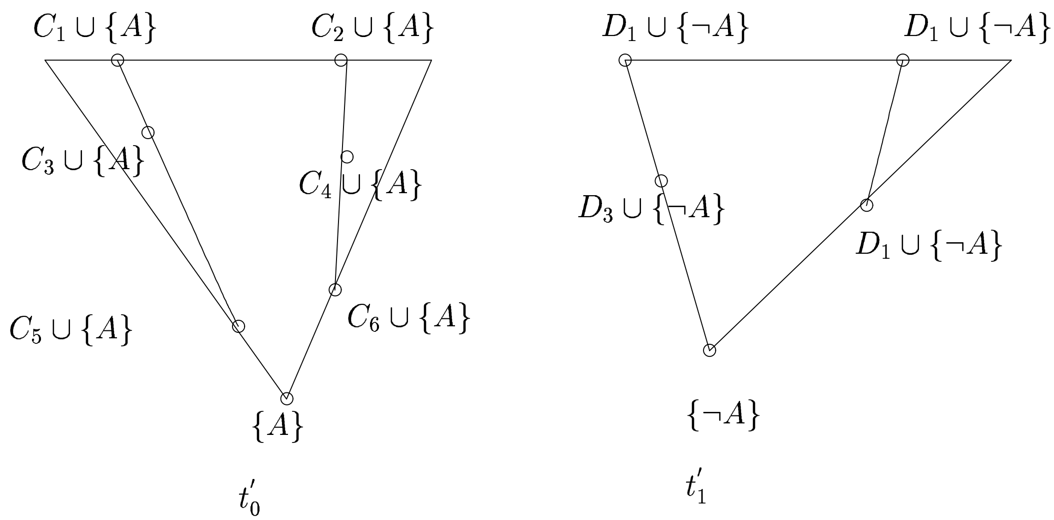

Figure 3: The trees t'_0 and t'_1 from Question 4

Both t_0 and t_1 have height at most n. If any of t_0 and t_1 are S-trees, i.e. all there leaves are in S we are done; that tree is a derivation tree of $\Box$ from S. So, let us assume that they are not. Then some leaves in t_0 are missing the A atom and some leaves in t_1 are missing $\neg A$. We add A to the leaves of t_0 that are missing A and to all their ancestors. We get the tree t'_0. We add $\neg A$ to all the leaves of t_1 that are missing $\neg A$ and to all their ancestors. We get the tree t'_1. This construction is shown in Figure 3.

The trees t'_0 and t'_1 are derivation trees with leaves in S and have height less than or equal to n. Finally, we unify the roots of these trees and get the derivation tree t from Figure 4.

Since the heights of t'_0 and t'_1 are less than or equal to n, t has height less than or equal to $n+1$.

Grading Criteria:

1. The $\Longleftarrow$ implication : 3 points
2. The $\Longrightarrow$ implication : 22 points

2.1. Using the induction method: 3 points
2.2. The basis: 3 points
2.3. The IH (getting t_0 and t_1) : 5 points
2.4. Getting t'_0 and t'_1 : 9 points
2.5. Getting t: 3 points

Question 5 (20 points)

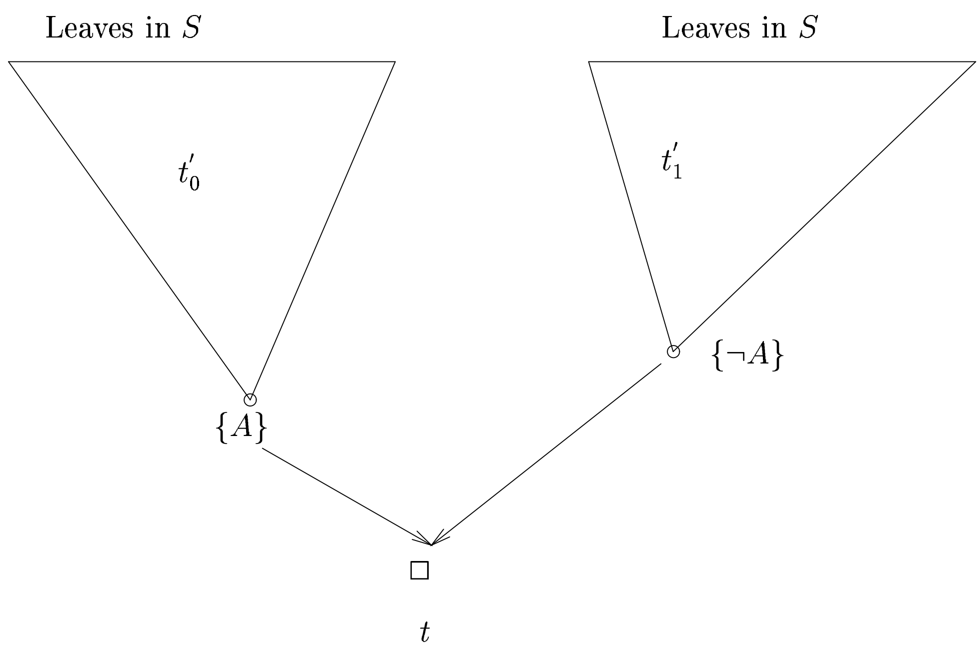

Figure 4: The trees t from Question 4

Find a CNF for $F = \neg[((A \vee B) \wedge C) \longleftrightarrow \neg(B \vee D)]$.

Show your work.

$F = \neg[((A \vee B) \wedge C) \longleftrightarrow \neg(B \vee D)]$

$\equiv \neg\{[((A \vee B) \wedge C) \longrightarrow \neg(B \vee D)] \wedge [\neg(B \vee D) \longrightarrow ((A \vee B) \wedge C)]\}$ $\longleftrightarrow$ elimination, line 1

$\equiv \neg\{[\neg((A \vee B) \wedge C) \vee \neg(B \vee D)] \wedge [\neg\neg(B \vee D) \vee ((A \vee B) \wedge C)]\}$ $\longrightarrow$ elimination, line 2

$\equiv \neg[\neg((A \vee B) \wedge C) \vee \neg(B \vee D)] \vee \neg[\neg\neg(B \vee D) \vee ((A \vee B) \wedge C)]$ De Morgan's law, line 3

$\equiv [\neg\neg((A \vee B) \wedge C) \wedge \neg\neg(B \vee D)] \vee [\neg\neg\neg(B \vee D) \wedge \neg((A \vee B) \wedge C)]$ De Morgan's law twice, line 4

$\equiv [((A \vee B) \wedge C) \wedge (B \vee D)] \vee [\neg(B \vee D) \wedge (\neg(A \vee B) \vee \neg C)]$ double negation elimination 3 times, De Morgan's once, line 5

$\equiv [(A \vee B) \wedge C \wedge (B \vee D)] \vee [(\neg B \wedge \neg D) \wedge ((\neg A \wedge \neg B) \vee \neg C)]$ De Morgan's twice, line 6

$\equiv [(A \vee B) \wedge C \wedge (B \vee D)] \vee [\neg B \wedge \neg D \wedge (\neg A \vee \neg C) \wedge (\neg B \vee \neg C)]$ distributivity, line 7

$\equiv [(A \vee B) \wedge C \wedge (B \vee D)] \vee [\neg B \wedge \neg D \wedge (\neg A \vee \neg C)]$ absorbtion

$\equiv (A \vee B \vee \neg B) \wedge (A \vee B \vee \neg D) \wedge (A \vee B \vee \neg A \vee \neg C) \wedge (C \vee \neg B) \wedge (C \vee \neg D) \wedge (C \vee \neg A \vee \neg C) \wedge (B \vee D \vee \neg B) \wedge (B \vee D \vee \neg D) \wedge (B \vee D \vee \neg A \vee \neg C)$ generalized distributivity, line 8

$\equiv (A \vee B \vee \neg D) \wedge (\neg B \vee C) \wedge (C \vee \neg D) \wedge (\neg A \vee B \vee \neg C \vee D)$ tautologies, line 9

Grading Criteria: You get credit up to the first line where an error occurred. At line 2, 1 point; at line 3, 3 points; at line 4, 5 points; at line 5, 7 points; at line 6, 10 points; at line 7, 12 points; at line 8, 14 points; at line 9, 17 points; no errors, 20 points.

Bonus (15 points)

Let F be a formula with atoms $P_1, \ldots, P_n$ and let S be the set of all clauses C with literals in the set $\{P_1, \neg P_1, P_2, \neg P_2, \ldots, P_n, \neg P_n\}$ that satisfy the relation $F \models C$. Prove that $S \equiv F$, i.e. every model of S is a model of F and viceversa.

Proof: We need to show that

(1) $\models_{\mathcal{A}} F \Longrightarrow \models_{\mathcal{A}} S$

and

(2) $\models_{\mathcal{A}} S \Longrightarrow \models_{\mathcal{A}} F$

We prove (1) first. Let $C \in S$. Since $F \models C$ and $\models_{\mathcal{A}} F$, $\models_{\mathcal{A}} C$. So, we conclude $\models_{\mathcal{A}} S$ because C is arbitary in S.

We now prove (2). Let $\models_{\mathcal{A}} S$. By the CNF Theorem, F has a CNF, $C_1 \wedge \ldots \wedge C_m \equiv F$. Since the atoms of F are in the set $\{P_1, \ldots, P_n\}$, $C_1, \ldots, C_m$ have atoms in the same set. Moreover, for all $1 \leq i \leq m$, $F \models C_i$. So, $C_1, \ldots, C_m \in S$. Since $\models_{\mathcal{A}} S$, $\models_{\mathcal{A}} C_1 \wedge \ldots \wedge C_m$. Then, $\models_{\mathcal{A}} F$, because $C_1 \wedge \ldots \wedge C_m \equiv F$.

Grading Criteria:

1. Listing (1) and (2) - 2 points
2. Proving (1) - 4 points
3. Proving (2) - 9 points