Lecture Notes for EGN 5422

Engineering Applications of Partial Differential Equations

Summer and Spring Semesters

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The current syllabus and the takehome exams are available from the course Blackboard site.

CONTENTS

Generic Syllabus: EGN 5422 or EEL 6935 ENGINEERING APPLICATIONS OF PARTIAL DIFFERENTIAL EQUATIONS

The 5000- and 6000-level sections are identical in content, but the 6000-level section is graded at a higher standard. Contact the Electrical Engineering Department if you wish to enroll in a 6000-level section.

Instructor: Prof. A. D. (Dave) Snider, ENB 246A, 813-974-4785, FAX: 813/974-5250 snider@usf.edu

Office hours (see Blackboard)

Text: *Partial Differential Equations: Sources and Solutions*, A. D. Snider, Dover Publications, Mineola NY, 2006, ISBN 0-486-45340-5, *required*. NOTE: Computer access (PC) is also required.

Catalog Description: Power series solutions for ordinary differential equations, Sturm-Liouville theory, special functions. Separation of variables for partial differential equations. Green's functions. Use of USFKAD software.

Course Prerequisites: MAP 2302 Differential Equations (solution methods for first order nonlinear and higher order linear equations)

The **lectures** can be viewed (by anyone at any time) at http://netcast.usf.edu/browse.php?page=Classes/engineering/snider/PartialDiffEqs

However, there is an error in the listings: there is NO movie "Lecture 48_3D Examples". Simply skip it.

Welcome to my classes, USF students!

You're taking a "web-only" class from me. That means you won't see me live; you'll watch lectures I taped in the past, on the course materials. I will be available in my office for consultation a few hours per week, and we can Skype. I'll answer your emails promptly, *unless they are addressed in the syllabus or at the course web site in Blackboard*, https://my.usf.edu. This site contains class announcements, documents, old and current exams, assignments, etc.

Think about this: you're taking a course from a professor whom you'll hardly ever see. So you don't have much chance for an "A" unless you

(1) buy the textbook and

(2) *follow the web procedures precisely*.

Be sure to read this long-winded syllabus entirely. Then print it out and stick it in the back of your textbook for future reference.

You may have misgivings about non-live lectures. Let me relate a few observations that I have made over the past years with this process.

First of all, this procedure has proved quite successful - to my surprise. I was leery about it at first, but I soon found out that the average student performance on tests was slightly better than it had been before, when I was lecturing live. I think the reasons for this are as follows:

1. You will never need to miss a lecture due to illness, conflicting appointments, being out of town, or simply being tired. The lectures are at the web site all semester long, 24/7.
2. You can re-watch any lecture, or part of a lecture, as many times as you need.
3. If you need to take a break (no student has ever fallen asleep during my lectures, of course!!!), you can stop the playback and resume when you return.

The lectures are numbered in order (except for lecture 51, Numerics, which can be watched anytime after you have mastered the software *and before you attempt the takehome*). They were taped during different semesters, so you will have to figure out the right pace to get you through them during the time allotted in your current semester. If you can't do this, please drop my class and re-take MTH 101, paying particular attention to the lessons on ratio and proportion.

Any *administrative* instructions that I give in the lectures are probably out of date. Consult your email and Blackboard weekly for updates from me on assignments, test dates, and procedures in general.

Now let me take one paragraph to try to talk you out of taking this course. Partial differential equations is a very difficult subject. In my book and lectures I have tried to make it as simple as possible; but still EGN 5422 is the hardest course I teach. Only about 50-60% of the students who take it make an A; you need to be very good at mathematics to achieve an A. Don't take this course if you can't afford a B on your transcript; if you need to know the subject, consider auditing the course for non-credit.

Be sure to read the document on accurate calculations. *Remember these tips in all your classes.* Instructions for using MATLAB to perform the integrations for the takehome assignment are covered in this document.

Lecture notes replicating everything that appears on the blackboard during the lectures can be purchased from

> Pro-Copy
> 5219 E. Fowler Ave.
> Tampa, Fl 33617
> procopy1@aol.com<mailto:procopy1@aol.com>
> (813) 988-5900

They can also be downloaded from Blackboard. (Ignore notes in the .xbk format.) If you download, print out the lecture notes before you watch the lecture, and have them in front of you.

The software USFKAD can be downloaded from Blackboard. LaTeX software is required to run USFKAD. LaTeX (the MiKTeX package) is available at www.miktex.org . Download LaTeX during the first week and read the notes at Blackboard for installing it. *Experience has shown that if you wait until midsemester to start working with LaTeX, you will sabotage any chance of success in this course*. Test your installation by trying to open and read the article CompPhys4.tex .

Requirements and Assessment:

1. Each student must email Prof. Snider with the following data: Last name: ______ First name: _______ Class: EGN 5422, by (see Blackboard). I will not acknowledge these emails individually. In one week you will receive an acknowledgement, by email, from me (Dr. Snider) that you are in his class email address list; if you do not receive this acknowledgement, email me again – until I acknowledge receipt. **Thereafter each student is liable for all email notices concerning the class from Prof. Snider.** Students who wish to use different email boxes should email this data from each box. Do not use one email box to request mail to a different box.

2. Each student must sign a copy of the final page of this syllabus as indicated below and submit it to Dr. Snider by (see Blackboard). You are not officially enrolled in the class until you have turned in a signed syllabus. **Postal-mail a hard copy** to Dr. A D Snider, Dept. of Electrical Engineering, University of South Florida, 4202 East Fowler Avenue ENB 118, Tampa FL 33620; or put a copy in my EE Department mailbox. **Email is not acceptable**. I will not acknowledge receipt of these syllabi individually, but they can be regenerated later in the semester if a problem arises.

3. Certain homework problems will be recommended to the students, but not graded. You should regard the old tests as a prime source of homework problems; work them during the semester as the particular topic is covered in the lectures. Once you have mastered the USFKAD software, you can use it to confirm your answers.

4. A midterm examination (time and place to be arranged, based on section 5.2 only) and a final will be given. Note that answers to all the problems in Section 5.2 are given in Blackboard's "Course Documents" A time and place on the Tampa campus will be arranged for these tests, but they can be taken at remote sites and more convenient times (within limits) if a proctor agreement is worked out; I'll notify you.

Additionally a takehome test, based lecture #51 "Numerics," will be assigned. The takehome is heavily computational and will be computer graded with no partial credit, but you will be allowed four attempts (with, however, different numerical parameters each time). You will submit your answers by email to the TA, who will shortly respond with your score, a tabulation of your incorrect answers, and the correct answers for the parameters you used. Your first attempt is assessed at (only) 5%, the second at 55%, the third at 20%, and the fourth at 20%. Each attempt has a deadline. If you miss a deadline, your subsequent submission will also count as the missed submission. If you stop submitting at any time, your last submission will count for the subsequent ones. The takehome tests will be available at Blackboard, and they contain more detailed instructions.

Time permitting, you will be asked to present an oral defense of your takehome and exam solutions.

Your final grade will be a weighted average of the midterm (20%), the takehome (40%), and the final (40%).

I recommend that you take a timed midterm and a timed final from Blackboard for practice. These tests are *open*-book, *closed* notes; you are *not* permitted to bring old tests to the exams. (I'm not trying to be tough; experience has shown that old tests are counterproductive rather than helpful.)

5. An "incomplete" grade will be awarded if either the email, syllabus signoff, midterm, takehome, or final are not submitted. Incompletes can be made up following USF policies; you are on your own to figure out these policies. However you need to know that I am retired, and only work as an adjunct faculty, so I cannot guarantee my availability after the semester is over. If you need a flexible, cooperative professor instead of a cranky old curmudgeon, you are advised to take another class.

Please mail a copy of **this page** to Dr. Snider.

Academic Dishonesty - It is not acceptable to copy, plagiarize or otherwise make use of the work of others in completing homework, project, exam or other course assignments. The minimum penalty for doing so is an automatic zero on the assignment and an "F" in the course. I have read the syllabus for EGN 5422, Spring 2011, and agree to abide by its schedule and terms.

Print name: Sign name:

Accurate calculations and evaluation of integrals using MATLAB.

Tips on calculating:

Some tests in this course require a lot of calculation. Here are some pointers that you may not have picked up yet:

1. If you require a certain number of significant digits in a final answer, you must maintain *more* digits in the intermediate calculations. For instance consider the addition 1/3 + 1/3 = 2/3. The three-significant-digit expression for the answer, 2/3, is .667 . But if you first round the addends to three significant digits you get .333 + .333 = .666, which is *not correct* to 3 digits. A good rule of thumb is to retain at least two more digits than required, in *all* intermediate calculations.

2. Always introduce symbols (letters) when you evaluate a formula, especially on a calculator. For instance suppose you wanted to add 436 and 578 on a calculator. The worst way to do it is to enter 436, press enter, press +, enter 578, press enter, and press =. Because: suppose you get an answer you know is wrong, like -23. You don't know whether you entered the 436 wrong, or the 578, or the + sign. You have to reenter all the data again. The smart way is to let A equal 436, let B equal 578, and call for A+B. If the answer is absurd, you can recall A, B, and the formula; and you can correct only the one that's wrong. This is particularly significant when you're dealing with high-digit numbers, and complicated formulas that may require parentheses.

Remember these tips in all your classes.

3. To add an infinite series. Unless you know a lot about the series, there is no sure way to tell when you have achieved a specified accuracy. Here's what conservative engineers do in practice:

Add up a selected number of terms - 5, 10, whatever. Then double the number of terms and add again. Compare the second total to the first; if they agree to within the prescribed accuracy, you're probably OK. (Naturally, take the second total as your answer.) If the required accuracy is expressed as a percentage, the difference between the two totals must be less than the specified percentage of the final total.

If they don't agree sufficiently well, double the number of terms and add again, and so on until you get agreement in two consecutive sums, to within the specified accuracy.

If you can't get agreement with your resources, you're going to need a professional mathematician.

4. Note: if you don't know MATLAB, you still should be able to use it to evaluate integrals; just open it (click on the MATLAB icon) and notice how functions are entered into the "@" statements below - multiplication, sines, exponentials, division, CORRECT NUMBER OF PARENTHESES.

I. ***One-dimensional integrals***. To integrate $\int_{1.234}^{5.678} [e^{-2x^2} \cos(4x+7) - \frac{2x+1}{(x+3)^2 (x+1)(x+2)}] dx$:

in MATLAB type

format long (Enter)

a=2 (Enter)

b=4 (Enter)

c=7 (Enter)

d=2 (Enter)

e=1 (Enter)

f=3 (Enter)

g=1 (Enter)

h=2 (Enter)

lower=1.234 (Enter)

upper=5.678 (Enter)

accuracy = 1e-8 (Enter)   (Of course you may want more or less accuracy.)

Now enter the formula for the integrand, REGARDING THE VARIABLE OF INTEGRATION (x) AS A MATRIX. That means inserting the period mark (.) in front of all multiplication signs *, division signs /, and exponentiation carats ^ involving x.

integrand=@(x)(exp(-a*x.^2).*cos(b.*x+c) - (d.*x+e)./((x+f).^2.*(x+g).*(x+h))) (Enter)

(You'll probably get some typo error message here; retype carefully, counting parentheses and inserting dots. An error message like

"??? Error using ==> mtimes

Inner matrix dimensions must agree."

means you left out some dots. An error message like

" Error: Expression or statement is incorrect--possibly unbalanced (, {, or [."

usually means your parentheses are wrong. *If you change the values of any of the parameters,* ***re****enter the integrand=@... statement*.) Now type

Q=quadl(integrand, lower, upper, accuracy) (Enter)

I got an answer of Q = -0.037041299610442 .

II. ***Two-dimensional integrals***. To integrate $\int_4^5 \int_6^7 (u^2 + uv)\, dv du$:

in MATLAB type

format long (Enter)

lowerv=6 (Enter)

upperv=7 (Enter)

loweru=4 (Enter)

upperu=5 (Enter)

accuracy = 1e-8 (Enter)

integrand=@(u,v)(u.^2+u.*v) (Enter)

Q= dblquad(integrand, loweru,upperu,lowerv,upperv, accuracy, @quadl) (Enter)

I got Q = 49.583333333333329

III. ***Three-dimensional integrals***. To integrate $\int_2^3 \int_4^5 \int_6^7 uvw\, dw\, dv\, du$:

in MATLAB type

format long (Enter)

loww=6 (Enter)

highw=7 (Enter)

lowv=4 (Enter)

highv=5 (Enter)

lowu=2 (Enter)

highu=3 (Enter)

accuracy = 1e-8 (Enter)

integrand=@(u,v,w)(u.*v.*w) (Enter)

Q= triplequad(integrand, lowu,highu,lowv,highv,loww,highw,accuracy, @quadl) (Enter)

I got Q = 73.124999999999986 .

To be conservative, try out your code with integrands that you can do by hand, to check.

Some Solutions

EGN 5422 Solutions

$\underline{1.1}$ 1.a. $y = c_1 e^{x} + c_2 e^{-2x}$

b. $y = c_1 e^{-x/2} + c_2 x e^{-x/2}$

c. $y = e^{-3/10 x}\left(c_1 \cos x/10 + c_2 \sin x/10\right)$

d. $y = e^{x/\sqrt{2}}\left(c_1 \cos x/\sqrt{2} + c_2 \sin x/\sqrt{10}\right)$
$\quad + e^{-x/2}\left(c_3 \cos x/\sqrt{2} + c_4 \sin x/\sqrt{10}\right)$

f. ~~e~~. $y = \dfrac{c_1}{\sqrt{x}} + \dfrac{c_2}{x}$

e. ~~f~~. $y = c_1 x + c_2 x^2$

g. $y = c_1/x + c_2 x + c_3 \cos(3 \log x) + c_4 \sin(3 \log x)$

3b. $A = \pm\sqrt{b_0^2 + b_1^2}$

$$\varphi = \frac{1}{2}\cos^{-1}\left(\frac{b_0^2 - b_1^2}{b_0^2 + b_1^2}\right) - a$$

or

$$\varphi = \tan^{-1}\left(\frac{-b_1}{b_0}\right) - a$$

3c. $c_1 = A \cos \varphi; \quad c_2 = -A \sin \varphi$

$\underline{1.2}$ 1a. One sol. is $y = x^{1/2} J_{\frac{1}{m+2}}\left(\frac{2}{m+2} x^{\frac{m+2}{2}}\right)$

b. One sol. is $y = x^{-1/2} J_{1/2}(2x)$

c. One sol. is $y = x^{1/2} J_{1/2}(x)$

$\underline{1.3}$ 4a. $f''(x) = 2\,\delta(x) + \begin{cases} -1 & x < 0 \\ 1 & x > 0 \end{cases}$

11. $f(x) = \dfrac{\cos x}{x^2(x^2-1)} + c_1 \delta(x) + c_2 \delta'(x) + c_3 \delta(x+1) + c_4 \delta(x-1)$

Answers

§ 1.2

~~BR~~ #1a. $x(t) = \frac{t-a}{b}$ $\quad \frac{dx}{dt} = \frac{1}{b}$ $\quad \frac{d^2x}{dt^2} = 0$ $\qquad t = a + bx$

From Eq. (8), $y' = Y'/\tfrac{1}{b} = bY'$

Eq (9), $y'' = \frac{Y''}{(1/b)^2} - \frac{Y' \cdot [0]}{[\ ]^3} = b^2 Y''$

$$y'' + A(a+bx)^r y' + B(a+bx)^s y = 0$$

transforms to

$$b^2 Y'' + A t^r b Y' + B t^s Y = 0$$

? no such problem

b. Choose A, B, r, s, a, and b so that we can identify:

$y'' + xy = 0$ with $y'' + A(a+bx)^r y' + B(a+bx)^s y = 0$

and $Y'' - tY = 0$ with $b^2 Y'' + A t^r b Y' + B t^s Y = 0$.

Reason as follows:

no y' term: $A = 0$

~~exponent of y coefficient is 1: s=1~~

match up: $xy = B(a+bx)^s y$

$-tY = B t^s Y$

$b^2 = 1$

$\Rightarrow B = -1,\ s = 1,\ b = -1,\ a = 0$

So $\boxed{x = -t}$

? no such problem

c. Identify $y'' - xy' - y = 0$ with $y'' + A(a+bx)^r y' + B(a+bx)^s y = 0$

and $Y'' - 2tY' - 2Y = 0$ with $b^2 Y'' + A t^r b Y' + B t^s Y = 0$.

From the original equation, $-y = B(a+bx)^s y \Rightarrow s = 0$ and $B = -1$.

From the second equation, $-2Y = B t^s Y \Rightarrow B = -2$ $\boxed{\text{contradicts } B = -1}$.

So rewrite the second equation as

$$Y'' + \frac{A t^r}{b} Y' + B \frac{t^s}{b^2} Y = 0$$

Now match: $-2Y = \frac{B t^s}{b^2} Y$, $s = 0$, $B = -1$; so $b = \pm \frac{1}{\sqrt{2}}$.

§1.2

~~2.3~~ #3 $x = e^t \qquad \frac{dx}{dt} = e^t = \frac{d^2x}{dt^2}$

$$y' = Y'/e^t \qquad y'' = \frac{Y''}{(e^t)^2} - \frac{Y'(e^t)}{(e^t)^3}$$

$$0 = a_2 x^2 y'' + a_1 x y' + a_0 y = a_2 e^{2t}\left[Y'' e^{-2t} - Y' e^{-2t}\right] + a_1 e^t \frac{Y'}{e^t} + a_0 Y$$

$$= a_2 Y'' + [a_1 - a_2] Y' + a_0 Y \quad \left(\begin{matrix}\text{constant}\\ \text{coefficients}\end{matrix}\right)$$

#5 $x = \cos\phi \qquad \frac{dx}{d\phi} = -\sin\phi \qquad \frac{d^2x}{d\phi^2} = -\cos\phi$

$$y' = Y'/(-\sin\phi) \quad y'' = \frac{Y''}{(-\sin\phi)^2} - \frac{Y'(-\cos\phi)}{(-\sin\phi)^3}$$

$$(1-x^2)y'' - 2xy' - \lambda y$$

$$= \sin^2\phi\left[\frac{Y''}{\sin^2\phi} - \frac{\cos\phi}{\sin^3\phi}Y'\right] + \frac{2\cos\phi\, Y'}{\sin\phi} - \lambda Y = 0$$

$$= Y'' + \frac{\cos\phi}{\sin\phi}Y' - \lambda Y = 0$$

or

$$\sin^2\phi\, Y'' + \cos\phi\sin\phi\, Y' - \lambda\sin^2\phi\, Y = 0$$

~~2.3~~ #2a. On the left, $f' = 0$; on the right, $f' = [2\cos x]' = -2\sin x$.

§1.3

At $x=0$ f jumps by 2, so f' contains $2\,\delta(x)$.

$\therefore\ f(x) = (-2\sin x)\ H(x) + 2\,\delta(x)$

b. $f'(x) = -2\sin x\, H(x) + 2\cos x\, H'(x) = -2\sin x\, H(x) + \underbrace{2\cos x\, \delta(x)}_{= 2\cos 0\, \delta(x) = 2\,\delta(x)}$

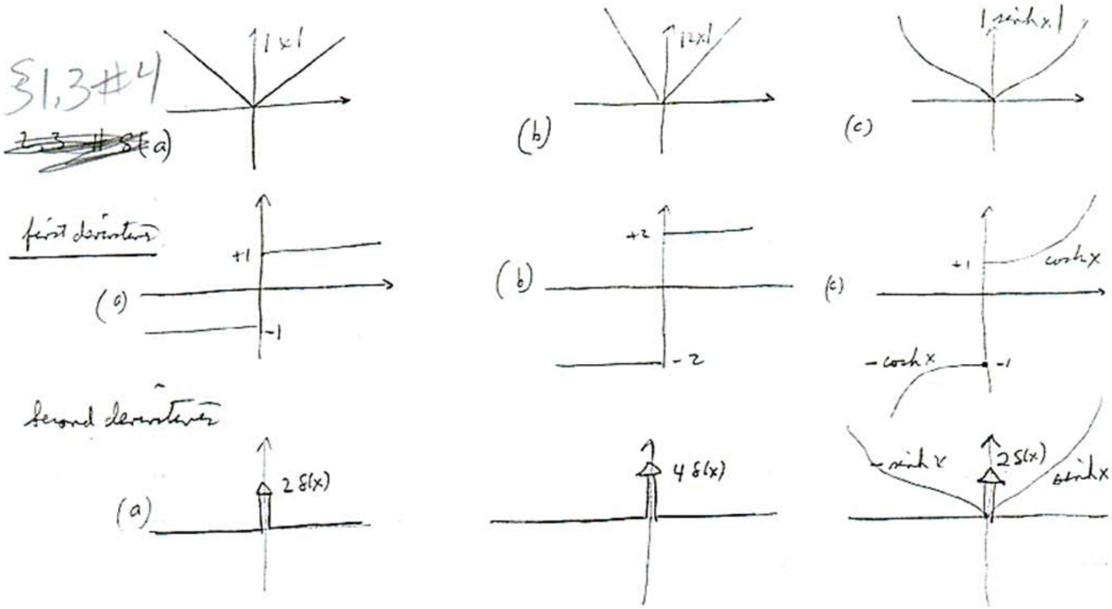
c. $$\int_{-\infty}^{\infty} f g'\,dx = \int_0^{\infty} 2\cos x\, g'\,dx = 2\cos x\, g(x)\Big|_0^{\infty} - \int_0^{\infty} (-2\sin x)\, g(x)\,dx$$

$$= 2\cos\infty\, \overset{0}{\overset{\shortparallel}{g(\infty)}} - 2g(0) + \int_0^{\infty} 2\sin x\, g(x)\,dx$$

$$= \int_{-\infty}^{\infty} [2\sin x\, H(x) - 2\,\delta(x)]\, g(x)\,dx.$$

$$= -\int_{\infty}^{\infty} f' g\,dx, \qquad (QED).$$

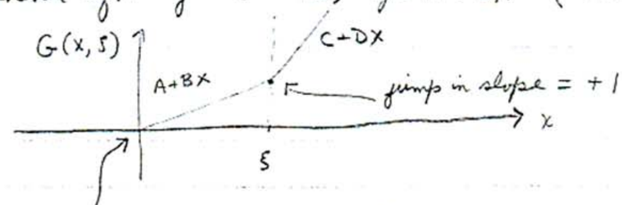
§1.3 #4



$y'' = 2\delta(x)$ $\qquad$ $y'' = 4\delta(x)$ $\qquad$ $y'' = 2\delta(x) + |\sinh x|$

§1.4

~~1.4~~ #1(a) Homogeneous eq.: $y'' = 0 \implies y = A + Bx \quad (C + Dx)$



Here $y = y' = 0$. So $A = 0$, $B = 0$.

$$\left.\begin{array}{lll} \text{at } x=\xi, & C + D\xi = A + B\xi = 0 \\ \text{at } x=\xi, & D = B + 1 = 1 \end{array}\right\} \text{ so } C + 1\cdot\xi = 0,\ C = -\xi$$

$$G(x,\xi) = \begin{cases} 0 & x < \xi \\ -\xi + 1\cdot x & x > \xi \end{cases}$$

#~~1~~3. For $x < \xi$, $G = \frac{-1}{2K} e^{K(x-\xi)}$, $\frac{\partial^2 G}{\partial x^2} = \frac{K^2}{2K} e^{K(x-\xi)} = K^2 G$: checks.

For $x > \xi$, $G = \frac{-1}{2K} e^{-K(x-\xi)}$, $\partial^2 G/\partial x^2 = \frac{K^2}{2K} e^{-K(x-\xi)} = -K^2 G$: checks.

Jump in slope at $x = \xi$: $\frac{\partial G}{\partial x}(x=\xi+) - \frac{\partial G}{\partial x}(x=\xi-) = -\frac{(-K)}{2K} e^0 - \left(-\frac{K}{2K} e^0\right) = \frac{2K}{2K} = 1$: checks.

p 15

§2.1

~~#1~~ #2 $\Gamma(j+p+1) = (j+p)\Gamma(j+p) = (j+p)(j+p-1)\Gamma(j+p-1)$

$= (j+p)(j+p-1)\cdots(1+p)\Gamma(1+p)$. now divide.

#3 $(2+p)(4+p)(6+p)\cdots(2j+p) = \overbrace{2\cdot 2\cdot 2\cdots 2}^{j \text{ factors}} \underbrace{(1+\frac{p}{2})(2+\frac{p}{2})(3+\frac{p}{2})\cdots(j+\frac{p}{2})}_{\frac{\Gamma(j+p/2+1)}{\Gamma(p/2+1)}}$ by previous problem

$= 2^j$

#4 As in #2, $\Gamma(m+j+p+1) = (m+j+p)\Gamma(m+j+p)$

$= (m+j+p)(m+j+p-1)\cdots(m+p)\Gamma(m+p)$.

§2.3

~~#1~~ # 2

$y'' = xy \qquad = 0 \qquad y(0) = c_0$

$y''' = xy' + y \qquad = c_0 \qquad y'(0) = c_1$

$y'''' = xy'' + y' + y' = xy'' + 2y' \qquad = 2c_1$

$y^{(5)} = xy''' + y'' + 2y'' = xy''' + 3y'' \qquad = 0$

$y^{(6)} = xy'''' + y''' + 3y''' = xy^{(4)} + 4y^{(3)} \qquad = 4c_0$

$y^{(7)} = xy^{(5)} + 5y^{(4)} \qquad = 10c_1$

$y^{(8)} = xy^{(6)} + 6y^{(5)} \qquad = 0$

$y^{(9)} = xy^{(7)} + 7y^{(6)} \qquad = 28c_0$

$y^{(10)} = xy^{(8)} + 8y^{(7)} \qquad = 80c_1$

$y(x) = c_0 + c_1 x + \frac{c_0}{3!}x^3 + \frac{2c_1}{4!}x^4 + \frac{4c_0}{6!}x^6 + \frac{10c_1}{7!}x^7 + \frac{28c_0}{9!}x^9 + \frac{80}{9!}x^{10}$

$+ \cdots$

§2.4

#3 Indicial equation is $r^2+(p_{-1}-1)r+q_{-2}=0$ (Eq. (18))

(a) $y''-\frac{1}{2x}y'+\left(\frac{1}{2x}-\frac{5}{2x^2}\right)y=0$ $\quad p_{-1}=-1/2 \quad q_{-2}=-5/2$

$$r^2+\left(-\tfrac{1}{2}-1\right)r+\left(-\tfrac{5}{2}\right)=r^2-\tfrac{3}{2}r-\tfrac{5}{2}=0$$

$$r=\frac{\frac{3}{2}\pm\sqrt{\frac{9}{4}+\frac{20}{2}}}{2}=\frac{\frac{3}{2}\pm\frac{7}{2}}{2}=\frac{5}{2},-1$$

$y_1(x)=x^{5/2}+O(x^{7/2})$ $\quad y_2(x)=x^{-1}+O(x^0)$

(b) $y''+\left(1-\frac{3}{x}\right)y'+\frac{3}{x^2}y=0$ $\quad p_{-1}=-3 \quad q_{-2}=3$

$r^2+(-3-1)r+3=0 \quad =r^2-4r+3 \quad r=1,\ r=3$

$y_1(x)=x^3+O(x^4)$ $\quad y_2(x)=x+O(x^2)+A(\log x)y_1(x)$

possibly $A=0$

(c) $(x^2-1)y''+\cdots$

not zero at $x=0$. So $x=0$ is not a singular point.

$y(x)=c_0+c_1x+O(x^2)$

(d) $y''+\frac{1}{x}y'-y=0$ $\quad p_{-1}=1 \quad q_{-2}=0$

$r^2+(1-1)r+0=r^2=0 \quad r=0,0$

$y_1(x)=1+O(x)$ $\quad y_2(x)=1+O(x)+A(\log x)y_1(x)$

$A\neq 0$.

§2.4

~~HW3~~ #8 $\overbrace{(1-x^2)}^{(1-x)(1+x)} y'' - 2xy' - \lambda y = 0 \qquad t = x-1,\ x = t+1,\ \frac{dx}{dt} = 1,\ \frac{d^2x}{dt^2} =$

$\Rightarrow$

$$-t(t+2)Y'' - 2(t+1)Y' - \lambda Y = 0 \quad = [-t^2 - 2t]Y'' + [-2t-2]Y' - \lambda Y$$

$$Y'' + \frac{2(t+1)}{t(t+2)}Y' + \frac{\lambda}{t(t+2)}Y = 0$$

$$p_{-1} = \lim_{t\to 0} tp(t) = \lim_{t\to 0} \frac{2(t+1)}{(t+2)} = 1$$

$$q_{-2} = \lim_{t\to 0} t^2 q(t) = \lim_{t\to 0} \frac{\lambda t}{t+2} = 0$$

$$r^2 + (p_{-1}-1)r + q_{-2} = r^2 + (1-1)r + 0 = r^2 = 0 \qquad r = 0, 0$$

$$Y_1(t) = \sum_{n=0}^{\infty} c_n t^n \qquad Y_1' = \sum n c_n t^{n-1} \qquad Y_1'' = \sum n(n-1)c_n t^{n-2}$$

$$-\sum_{n=0}^{\infty} n(n-1)c_n t^n - \sum_{0}^{\infty} n(n-1)2c_n t^{n-1} - \sum_{0}^{\infty} 2n c_n t^n - \sum_{0}^{\infty} 2n c_n t^{n-1} - \sum_{0}^{\infty} \lambda c_n t^n =$$

$$m=n \qquad m=n-1,\ n=m+1 \qquad m=n \qquad m=n-1,\ n=m+1 \qquad m=n$$

$$0 = -\sum_{m=0}^{\infty} m(m-1)c_m t^m - \sum_{m=-1}^{\infty} (m+1)m\,2c_{m+1} t^m - \sum_{m=0}^{\infty} 2m c_m t^m - 2\sum_{m=-1}^{\infty} (m+1)c_{m+1} t^m - \sum_{m=0}^{\infty} \lambda c_m t^m$$

$$m=-1: -(-1+1)(-1)2c_0 - 2(-1+1)c_0 = 0 = 0\cdot c_0 \qquad (c_0 \text{ arbitrary})$$

$$m \geq 0: \ -m(m-1)c_m - (m+1)m\,2c_{m+1} - 2mc_m - 2(m+1)c_{m+1} - \lambda c_m = 0$$

$$c_m[-m^2 + m - 2m - \lambda] + c_{m+1}[-2m^2 - 2m - 2m - 2] = 0$$

$$c_{m+1} = -\frac{m^2 + m + \lambda}{2m^2 + 4m + 2} c_m = -\frac{m(m+1) + \lambda}{2(m+1)^2} c_m$$

$$Y_1(t) = c_0\left[1 - \frac{\lambda}{2}t + \frac{2+\lambda}{8}t^2 - \frac{6+\lambda}{18}t^3 + \frac{12+\lambda}{32}t^4 + \cdots\right]$$

$$y_1(x) = c_0\left[1 - \frac{\lambda}{2}(x-1) + \frac{2+\lambda}{8}(x-1)^2 - \frac{6+\lambda}{18}(x-1)^3 + \frac{12+\lambda}{32}(x-1)^4 + \cdots\right]$$

$$y_2(x) = \sum_{n=0}^{\infty} d_n (x-1)^n + A\,(\log x)\, y_1(x)$$

$\neq 0$

§2.5

~~4.2~~ #8 $a = \frac{1}{2}$ $b = i\,\frac{2}{3}$ $c = \frac{3}{2}$ $n = \frac{1}{3}$

$$x^{1/2} J_{\pm 1/3}\left(i \frac{2}{3} x^{3/2}\right)$$

$$y = c_1 \sqrt{x}\, I_{1/3}\left(\frac{2}{3}x^{3/2}\right) + c_2 \sqrt{x}\, I_{-1/3}\left(\frac{2}{3}x^{3/2}\right)$$

~~4.4 #1~~ $x y'' + 3y' + xy = 0$ $y(1) = 0$ $y'(1) = 1$

2.5 #10

Eq. (18, 19): $x y'' - [(2a-1)]\, y' + \left[b^2c^2x^{2c-1} + \frac{(a^2 - \nu^2c^2)}{x}\right] y = 0$

$-(2a-1) = 3 \Rightarrow a = -1$ $2c - 1 = 1 \Rightarrow c = 1$ $b^2c^2 = 1 \Rightarrow b = 1$

$a^2 - \nu^2 c^2 = 0 \Rightarrow \nu^2 = a^2 \Rightarrow \nu = \pm 1$

$$y = c_1 \frac{1}{x} J_1(x) + c_2 \frac{1}{x} Y_1(x)$$

$$y(1) = c_1 J_1(1) + c_2 Y_1(1) = .4401\, c_1 + (-.7812)\, c_2 = 0$$

$$y'(1) = -\frac{c_1}{x^2} J_1(x) + \frac{c_1}{x} J_1'(1) - \frac{c_2}{x^2} Y_1(x) + \frac{c_2}{x} Y_1'(x)$$

$$= -c_1 J_1(1) + c_1\left[J_0(1) - \frac{1}{1} J_1(1)\right] - c_2 Y_1(1) + c_2\left[Y_0(1) - \frac{1}{1} Y_1(1)\right]$$

$$= c_1\left[J_0(1) - 2J_1(1)\right] + c_2\left[Y_0(1) - 2Y_1(1)\right]$$

since $c_1 J_1(1) + c_2 Y_1(1) = 0$

$$= c_1 J_0(1) + c_2 Y_0(1) = .7652\, c_1 + .0883\, c_2 = 1$$

$$c_1 = \frac{\begin{vmatrix} 0 & -.7812 \\ 1 & .0883 \end{vmatrix}}{\ldots} = \frac{.7812}{.6341} = 1.227$$

~~(4.4 #4)~~ $y(2) = c_1 J_{\frac{1}{2}}(2) + c_2 Y_{\frac{1}{2}}(2)$

§2.5 #10 $= (1.227 \times .5767 + .6913(-.1070))/2$

continued $= .3168$

§4.3

~~#8~~ #3 $\psi = \frac{1}{2}\int_{\tau=-\infty}^{t} \int_{\xi=x-v(t-\tau)}^{x+v(t-\tau)} h(\xi,\tau)\, d\xi\, d\tau$

$\frac{\partial h(\xi,\tau)}{\partial \xi} = h_1(\xi,\tau)$

$\frac{\partial h(\xi,\tau)}{\partial \tau} = h_2(\xi,\tau)$

$$\frac{\partial \psi}{\partial x} = \frac{1}{2}\int_{\tau=-\infty}^{t} \{h(x+v[t-\tau],\tau) - h(x-v[t-\tau],\tau)\}\, d\tau$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{2}\int_{\tau=-\infty}^{t} \{h_1(x+v(t-\tau),\tau) - h_1(x-v(t-\tau),\tau)\}\, d\tau$$

$$\frac{\partial \psi}{\partial t} = \frac{1}{2}\int_{\xi=x-v(t-t)}^{x+v(t-t)} h(\xi,t)\, d\xi \quad (=0)$$

$$+ \frac{1}{2}\int_{\tau=-\infty}^{t} \{h_1(x+v[t-\tau],\tau)\cdot(v) - h_1(x-v[t-\tau],\tau)(-v)\}\, d\tau$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{2}\{h_1(x+v[t-t],t)\cdot v - h_1(x-v[t-t],t)(-v)\} \; (=h(x,t))$$

$$+ \frac{1}{2}\int_{\tau=-\infty}^{t} \{h_1(x+v[t-\tau],\tau)\cdot(v^2) - h_1(x-v[t-\tau],\tau)(-v)^2\}\, d\tau$$

$$\frac{\partial^2 \psi}{\partial t^2} - v^2\frac{\partial^2 \psi}{\partial x^2} = h(x,t)$$

$\boxed{\S \cancel{1}4.3 \# 5}$ 1

(a) $\psi = \cos kx \sin kvt \qquad \frac{\partial^2\psi}{\partial x^2} = -k^2\psi \qquad \frac{\partial^2\psi}{\partial t^2} = -k^2v^2\psi$, So $\frac{\partial^2\psi}{\partial x^2} = \frac{1}{v^2}\frac{\partial^2\psi}{\partial t^2}$

Other: $\cos kx \cos kvt$, $\sin kx \sin kvt$, $\sin kx \cos kvt$.

(b) $\cos kx \sin kvt = \frac{e^{ikx}+e^{-ikx}}{2} \; \frac{e^{ikvt}-e^{-ikvt}}{2i}$

$= \frac{1}{4i}\left[e^{ik(x+vt)} - e^{-ik(x+vt)} \cancel{+} - e^{ik(x-vt)} + e^{-ik(x-vt)}\right]$

$= \frac{\sin k(x+vt)}{2} - \frac{\sin k(x-vt)}{2}$

(c) $\psi(x,0) = F(x) = (\cos kx)(\sin 0) \qquad F(x) = 0$

$\frac{\partial\psi}{\partial t}(x,0) = G(x) = kv(\cos kx)(\cos kv\cdot 0) \qquad G(x) = kv\cos kx$

$\boxed{\cancel{1}4.3 \# 4}$ Look at Eq. (9). We want only ~~the~~ functions of $(x+vt)$.
Let $G(\xi) = \frac{d}{d\xi}H(\xi)$; Then $\psi = \frac{F(x-vt)}{2} + \frac{F(x+vt)}{2} + \frac{H(x+vt)}{2v} - \frac{H(x-vt)}{2v}$

To cancel these, choose ~~$H(x-vt)$~~ $H(x-vt) = vF(x-vt)$

So answer: if $G(x) = \frac{d}{dx}vF(x) = vF'(x)$, then $\psi = F(x+vt)$

$\boxed{\S \cancel{1}4.6 \# 4}$ $\psi = A\log r + B \qquad A\log 1 + B = 1$

$r = \sqrt{(x-6)^2+(y-4)^2} \qquad A\log 4 + B = 0 \qquad B = 1 \quad A = -1/\log 4$

$\psi = -\frac{\log r}{\log 4} + 1$

(1a) $\dfrac{4\sin 3x \sinh 3y}{\sinh 3\pi}$ (1b) $\dfrac{3\sin 2x \sinh 2y}{\sinh 2\pi} - \dfrac{2\sin 3x \sinh 3y}{\sinh 3\pi}$

(1c) 0 (1d) $\dfrac{5}{8}\dfrac{\sin x \sinh y}{\sinh \pi} - \dfrac{5}{16}\dfrac{\sin 3x \sinh 3y}{\sinh 3\pi} + \dfrac{1}{16}\dfrac{\sin 5x \sinh 5y}{\sinh 5\pi}$

(1e) $\dfrac{\sin x \sinh y}{\cosh \pi}$ (1f) $\dfrac{3\sin 2x \sinh 2y}{2\cosh 2\pi} - \dfrac{2\sin 3x \sinh 3y}{3\cosh 3\pi}$

(2a) $\dfrac{\cos x \cosh y}{\cosh \pi}$ (2b) 1 (2c) 0 (2d) $\dfrac{3\cos 2x \cosh 2y}{\cosh 2\pi} - \dfrac{2\cos 3x \cosh 3y}{\cosh 3\pi}$

(2e) $\dfrac{3\cos 2x \cosh 2y}{2\sinh 2\pi} - \dfrac{2\cos 3x \cosh 3y}{3\cosh 3\pi}$

(5) $\psi = \sum_{n=1}^{\infty} a_n \sin nx \sinh ny \qquad \left.\dfrac{\partial\psi}{\partial y} - \psi\right|_{y=\pi} = \sum_{n=1}^{\infty} a_n \sin nx \left[n\cosh n\pi - \sinh n\pi\right] = x(\pi - x)$

so $a_n = \dfrac{\int_0^{\pi} x(\pi - x)\sin nx\,dx}{\underbrace{\int_0^{\pi}\sin^2 nx\,dx}_{\pi/2}\,\left[n\cosh n\pi - \sinh n\pi\right]}$

(6) $\psi = T_0$ (7) no solution

(8) $\psi = \sum_{n=1}^{\infty} a_n \sin nx \cosh ny \qquad \psi(x,\pi) = \sum a_n \sin nx \cosh n\pi = 1$

so $a_n = \dfrac{\int_0^{\pi} 1 \sin nx\,dx}{\underbrace{\int_0^{\pi}\sin^2 nx\,dx}_{\pi/2}\,\cosh n\pi}$

(9) $\psi = a_0 x + \sum_{n=1}^{\infty} a_n \sinh nx \cos ny \qquad \psi(\pi, y) = a_0\pi + \sum a_n \sinh n\pi \cos ny$

So $a_0 = \dfrac{\int_0^{\pi} 1\cdot 1\,dy}{\int_0^{\pi} 1^2\,dy\ \pi} = \dfrac{1}{\pi} \qquad a_n = \dfrac{\int_0^{\pi} 1\cos ny\,dy}{\int_0^{\pi}\cos^2 ny\,dy\ \sinh n\pi} = 0$

$\boxed{\psi = x/\pi}$

(11) $X = \sin nx \quad Y = c_1 \cosh ny + c_2 \sinh ny$

at $y=0$, $Y' = 2Y$ so $n c_2 = 2c_1$. Choose $Y = n \cosh ny + 2 \sinh ny$.

$$\psi = \sum_{n=1}^{\infty} a_n \sin nx \,[n \cosh ny + 2 \sinh ny]$$

$$\psi(x,\pi) = 1 = \sum a_n \sin nx \,[n \cosh n\pi + 2 \sinh n\pi] \quad \text{so } a_n = \frac{\int_0^\pi 1 \sin nx \, dx}{\underbrace{\int_0^\pi \sin^2 nx \, dx}_{\pi/2} \,[n \cosh n\pi + 2 \sinh n\pi]}$$

(12) $Y = 1, \cos ny \qquad X = a + bx, \quad c \cosh nx + d \sinh nx$

at $x=0$ $\quad X' = X \qquad X' = X$

$b = a \qquad nd = c$

$X = 1 + x \qquad X = n \cosh nx + \sinh nx$

$$\psi = a_0 (1+x) + \sum_{n=1}^{\infty} a_n \cos ny \,[n \cosh nx + \sinh nx]$$

at $x = \pi$, $\frac{\partial \psi}{\partial x} - 3\psi = 1 = a_0[1 - 3(1+\pi)] + \sum_{n=1}^{\infty} a_n \cos ny \,[n^2 \sinh n\pi + n \cosh n\pi - 3n \cosh n\pi - 3 \sinh n\pi]$

$$\text{so } a_0 = \frac{\int_0^\pi 1 \cdot 1 \, dx}{\int_0^\pi 1^2 \, dx \,[-2-3\pi]} \qquad a_n = \frac{\int_0^\pi 1 \cos ny \, dy}{\int_0^\pi \cos^2 ny \, dy \,[\;\cdots\;]} = 0$$

$$= \frac{\pi}{\pi[-2-3\pi]}$$

$$\boxed{\psi = \frac{-1}{2+3\pi}(1+x)}$$

(13) $$\psi = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{b} y \, \sinh \frac{n\pi}{b}(a - x)$$

$$\psi(0,y) = T_0 = \sum a_n \sin \frac{n\pi}{b} y \, \sinh \frac{n\pi}{b} a$$

$$\text{So } a_n = \frac{\int_0^b T_0 \sin \frac{n\pi y}{b} \, dy}{\underbrace{\int_0^b \sin^2 \frac{n\pi y}{b} \, dy}_{b/2} \, \sinh \frac{n\pi a}{b}}$$

§6.1 #1 ~~DE~~ $x^2y'' + xy' = \lambda y$ is equivalent to (14).
So general solutions are as indicated:

$$y(x) = \begin{cases} c_1 x^{\sqrt{\lambda}} + c_2 x^{-\sqrt{\lambda}} & \lambda > 0 \\ c_1 + c_2 \log x & \lambda = 0 \\ c_1 \cos(\sqrt{-\lambda}\log x) + c_2 \sin(\sqrt{-\lambda}\log x) & \lambda < 0 \end{cases}$$

$$y(1) = 0 \Rightarrow \begin{cases} c_1 = -c_2, & y = x^{\sqrt{\lambda}} - x^{-\sqrt{\lambda}} \\ c_1 = 0, & y = \log x \\ c_1 = 0 & y = \sin(\sqrt{-\lambda}\log x) \end{cases}$$

$$y(e) = 0 \Rightarrow \begin{cases} e^{\sqrt{\lambda}} - e^{-\sqrt{\lambda}} = 0 & \text{if } \lambda > 0 \text{ (impossible)} \\ \log e = 0 & \text{if } \lambda = 0 \text{ (impossible)} \\ \sin(\sqrt{-\lambda}) = 0 & \text{if } \lambda < 0 \text{ (O.K. if } \sqrt{-\lambda} = n\pi) \end{cases}$$

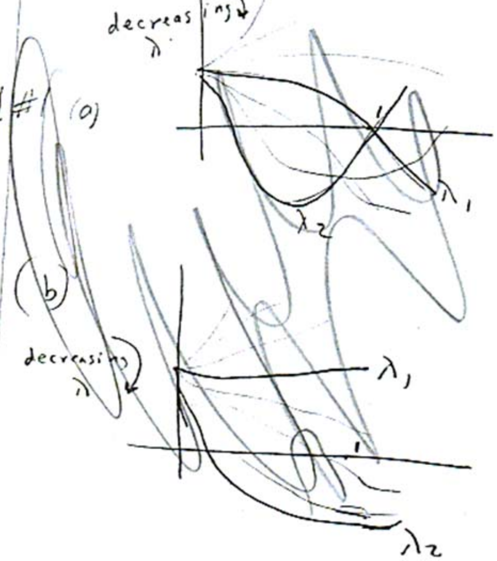
So $\lambda_n = -n^2\pi^2$; $y_n(x) = \sin(n\pi \log x)$; $n = 1, 2, 3, \ldots$
This is a transformed sine series; $\sin(n\pi\xi) = \sin(n\pi \log x)$ if $\xi = \log x$,

$$f(x) = \sum_{n=1}^{\infty} a_n \sin(n\pi \log x), \quad (1 < x < e), \quad a_n = \frac{\int_1^e f(x) \sin(n\pi \log x)\,\frac{dx}{x}}{\int_1^e \sin^2(n\pi \log x)\,\frac{dx}{x}}$$

is equivalent to

$$F(\xi) = \sum_{n=1}^{\infty} b_n \sin n\pi\xi, \quad 0 < \xi < 1, \quad b_n = \frac{\int_0^1 F(\xi) \sin n\pi\xi\, d\xi}{\int_0^1 \sin^2 n\pi\xi\, d\xi}$$

§6.1 #1 (a)



§6.2

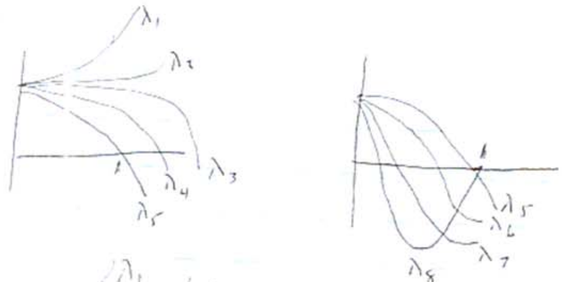
#6(a) $RHS = p(b)\left[f'(b)g(b) - f(b)g'(b)\right] - p(a)\left[f'(a)g(a) - f(a)g'(a)\right]$

Given $\left\{\begin{array}{l} \alpha f(a) + \beta f'(a) = 0 \\ \alpha g(a) + \beta g'(a) = 0 \end{array}\right\}$; if $\beta = 0$, $f(a) = g(a)$ and second bracket $= 0$
otherwise, $f'(a) = -\frac{\alpha}{\beta} f(a)$, $g'(a) = -\frac{\alpha}{\beta} g(a)$,
and second bracket $= \left[-\frac{\alpha}{\beta} f(a)g(a) + \frac{\alpha}{\beta} f(a)g(a)\right] = 0$,
Similarly for first bracket.

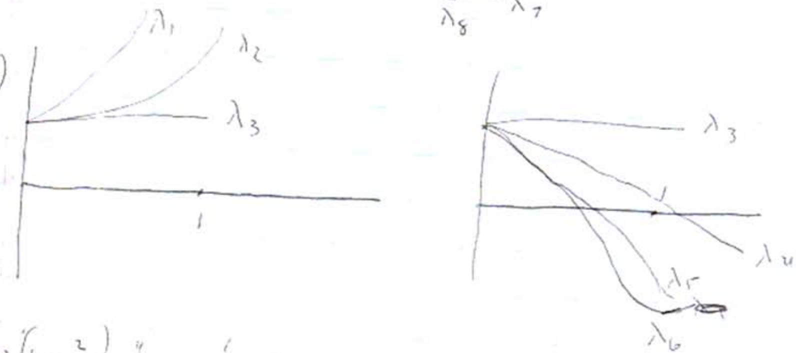
(b) Under these conditions $RHS = (\ldots) - (\text{same thing}) = 0$

§6.2 #1

(a)



(b)



#9(a) $(1-x^2)y'' - xy' + \lambda y = 0$

$$r(x) = \frac{1}{1-x^2}\exp\int^x \frac{-x}{1-x^2}\,dx = \frac{1}{1-x^2}\exp\left[\frac{\log(1-x^2)}{2}\right] = \frac{\sqrt{1-x^2}}{1-x^2} = \frac{1}{\sqrt{1-x^2}}$$

$\sqrt{1-x^2}\,y'' - \frac{x}{\sqrt{1-x^2}}\,y + \frac{\lambda}{\sqrt{1-x^2}}\,y = 0$ is the Sturm-Liouville form

(b) $y'' + xy' - y = 0$ $\quad r(x) = \frac{1}{1}\exp\int^x \frac{x}{1}\,dx = e^{x^2/2}$

$e^{x^2/2}y'' + xe^{x^2/2}y' - e^{x^2/2}y = 0$ is the S-L form

(c) $y'' - 3y' + 2y = 0$ $\quad r(x) = \frac{1}{1}\exp\int -3\,dx = e^{-3x}$

$e^{-3x}y'' - 3e^{-3x}y' + 2e^{-3x}y = 0$ is the S-L form

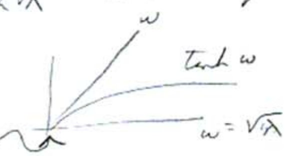
(d) $\left(1-\frac{x^2}{2}\right)y'' + xy' - y = 0$ $\quad r(x) = \frac{1}{1-x^2/2}\exp\int\frac{x}{1-x^2/2}\,dx = \frac{\exp\left[-\log\left(1-\frac{x^2}{2}\right)\right]}{1-x^2/2} = \frac{1}{(1-x^2/2)^2}$

$\frac{1}{1-x^2/2}\,y'' + \frac{x}{(1-x^2/2)^2}\,y' - \frac{1}{\left(1-\frac{x^2}{2}\right)^2}\,y = 0$ is the S-L form.

§6.3 #1 $\quad K=1$

Eq. (6) $\quad K \sinh\sqrt{\lambda} - \sqrt{\lambda}\cosh\sqrt{\lambda} = 0 \qquad \sqrt{\lambda} = \tanh\sqrt{\lambda}$ $\quad$ no solutions for $\lambda > 0$

both have slope = 1



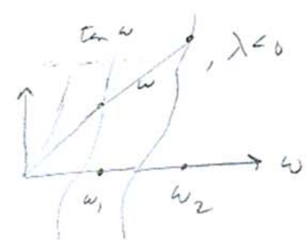
$K-1=0$ $\quad$ yes, there is a solution for $\lambda = 0$

So $X_0(x) = x \qquad Y_0(y) = y$

$K \sin\sqrt{-\lambda} - \sqrt{-\lambda}\cos\sqrt{-\lambda} = 0$

$(\omega = \tan\omega,$

$\omega = \sqrt{-\lambda})$



$X_n(x) = \sin\omega_n x \quad$ for $n = 1, 2, 3, \ldots$

$Y_n(y) = \sinh\omega_n y$

$$\psi(x,y) = \sum_{n=1}^{\infty} c_n \sin\omega_n x \sinh\omega_n y + c_0\, x y$$

$$\psi(x,2) = g(x)$$

$$c_{n>0} = \frac{\int_0^1 g(x)\sin\omega_n x\, dx}{\int_0^1 \sin^2\omega_n x\, dy \sinh 2\omega_n}$$

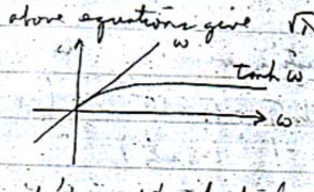
$$c_0 = \frac{\int_0^1 g(x)\, x\, dx}{\int_0^1 x^2\, dx\ (2)} = \frac{3}{2}\int_0^1 g(x)\, x\, dx$$

§6.3 another version.

#1 If $K=1$, eqs. (6) take the form

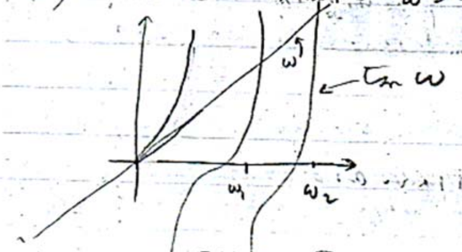
$$0 = \begin{cases} \sinh\sqrt{\lambda} - \sqrt{\lambda}\cosh\sqrt{\lambda} & (\lambda>0) \\ 0 & (\lambda=0) \\ \sin\sqrt{-\lambda} - \sqrt{-\lambda}\cos\sqrt{-\lambda} & (\lambda<0) \end{cases}$$

Thus $\lambda=0$ is an eigenvalue & the corresponding eigenfunctions is $(-x)$.
If $\lambda>0$ the above equations give $\sqrt{\lambda} = \tanh\sqrt{\lambda}$, which is impossible for $\lambda>0$:



(In fact, we don't need to check this! The eigenfunction $(-x)$ has no interior zeros, so it must be the first one!)

If $\lambda<0$, set $\lambda=-\omega^2$ and iterate $\omega = \tan^{-1}\omega$ to find the eigenvalues:



Eigenvalue	Eigenfunction	Y-factor
$\lambda_0=0$	$\phi_0(x)=x$ (drop the minus sign)	y
$\lambda_1=-\omega_1^2$	$\phi_1(x)=\sin\omega_1 x$	$\sinh\omega_1 y$
$\lambda_2=-\omega_2^2$	$\phi_2(x)=\sin\omega_2 x$	$\sinh\omega_2 y$
$\vdots$		

General solution $= a_0 x y + \sum_{n=1}^{\infty} a_n \sin\omega_n x \sinh\omega_n y$

Third boundary condition: $g(x) = a_0 x\cdot 2 + \sum_{n=1}^{\infty} a_n \sin\omega_n x \sinh\omega_n 2$

So

$$a_0 = \frac{\int_0^1 g(x)\, x\, dx}{\int_0^1 x^2\, dx \cdot 2}, \quad a_n = \frac{\int_0^1 g(x)\sin\omega_n x\, dx}{\int_0^1 \sin^2\omega_n x\, dx \,\sinh 2\omega_n}.$$

§6.3

#3 For $K = .1$, eq. (6) says

$$0 = \begin{cases} .1 \sinh\sqrt{\lambda} - \sqrt{\lambda}\cosh\sqrt{\lambda} \\ .1 - 0 \\ .1 \sin\sqrt{-\lambda} - \sqrt{-\lambda}\cos\sqrt{-\lambda} \end{cases} \quad \begin{array}{l} \sqrt{\lambda} = .1 \tanh\sqrt{\lambda} \\ \text{no sol'n} \\ \sqrt{-\lambda} = .1\tan\sqrt{-\lambda} \end{array}$$

Let $\sqrt{-\lambda} = \omega$. Then iterate

$$\omega_{n+1} = \tan^{-1}\omega_n/10.$$

For $K = -1$ the equations are

$$0 = \begin{cases} -1 \sinh\sqrt{\lambda} - \sqrt{\lambda}\cosh\sqrt{\lambda} & \text{(no solutions - both terms} < 0) \\ -1 - 1 \cdot \text{ (no sol'n)} \\ -1 \sin\sqrt{-\lambda} - \sqrt{-\lambda}\cos\sqrt{-\lambda} & \text{iterate } \omega_{n+1} = \tan^{-1}(-\omega_n), \\ & \lambda = -\omega^2. \end{cases}$$

~~2/15~~ First four values are;
(not completed)

8, ~~##~~ $u = X(x)Y(y)$ separates as always.

$$Y'' = \lambda Y \qquad Y'(0) = .2\, Y(0) \qquad Y'(1) = 0.$$

I think the boundary condition at $y = 1$ is simpler, so I'll write the general solution with y measured from the top:

$$Y(y) = \begin{cases} c_1 \cosh\sqrt{\lambda}(1-y) + c_2 \sinh\sqrt{\lambda}(1-y) & \lambda > 0 \\ c_1 + c_2(1-y) & \lambda = 0 \\ c_1 \cos\sqrt{-\lambda}(1-y) + c_2 \sin\sqrt{-\lambda}(1-y) & \lambda < 0. \end{cases}$$

Now $Y'(1) = 0$ implies $c_2 = 0$ in each case.

$$Y(y) = \begin{cases} \cosh\sqrt{\lambda}(1-y) \\ 1 \\ \cos\sqrt{-\lambda}(1-y) \end{cases}$$

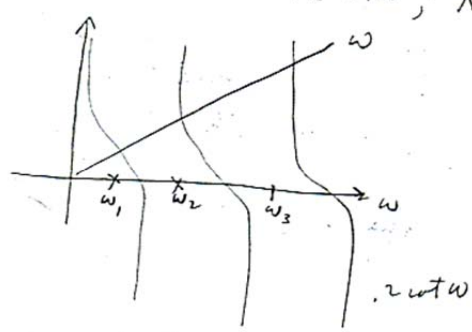
(# 8)

At $y=0$ we get

$$Y'(0) - .2\,Y(0) = 0 = \begin{cases} -\sqrt{\lambda}\,\sinh\sqrt{\lambda} - .2\cosh\sqrt{\lambda} \\ 0 - .2\times 1 \quad (\text{no solution}) \\ +\sqrt{-\lambda}\,\sin\sqrt{-\lambda} - .2\cos\sqrt{-\lambda} \end{cases}$$

$\sqrt{\lambda} = -.2\coth\sqrt{\lambda}$ $(\lambda>0)$ (impossible, since $\coth\sqrt{\lambda}>0$).

$\sqrt{-\lambda} = .2\cot\sqrt{-\lambda}$ or $\omega = .2\cot\omega$, $\lambda = -\omega^2$



So $Y_n(y) = \cos\omega_n(1-y)$ $\qquad X_n(x) = d_1\cosh\omega_n x + d_2\sinh\omega_n x$

$X_n(0) = 0 = d_1$

General soln: $\varphi(x,y) = \sum_{n=1}^{\infty} a_n \sinh\omega_n x\,\cos\omega_n(1-y)$

Bndry. cond. $g(y) = \sum_{n=1}^{\infty} a_n \sinh\omega_n(1)\,\cos\omega_n(1-y)$

$$\therefore\ a_n = \frac{\int_0^1 g(y)\cos\omega_n(1-y)\,dy}{\int_0^1 \cos^2\omega_n(1-y)\,dy\ \sinh\omega_n}$$ (weight function is unity)

(§6.3) 9. $\psi = X(x)\,Y(y)$ x-direction first.

$$X(x) = \begin{cases} c_1 \cosh\sqrt{\lambda}\,x + c_2 \sinh\sqrt{\lambda}\,x & (\lambda>0) \\ c_1 + c_2 x & (\lambda=0) \\ c_1 \cos\sqrt{-\lambda}\,x + c_2 \sin\sqrt{-\lambda}\,x & (\lambda<0) \end{cases}$$

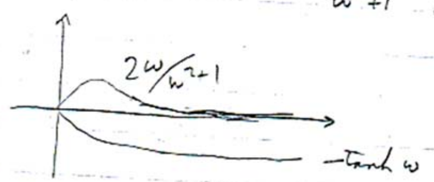
$X'(0) = X(0)$

$$\begin{cases} \sqrt{\lambda}\,c_2 = c_1 & X(x) = \sqrt{\lambda}\cosh\sqrt{\lambda}\,x + \sinh\sqrt{\lambda}\,x \\ c_2 = c_1 & X(x) = 1+x \\ \sqrt{-\lambda}\,c_2 = c_1 & X(x) = \sqrt{-\lambda}\cos\sqrt{-\lambda}\,x + \sin\sqrt{-\lambda}\,x \end{cases}$$

$X'(1) = -X(1)$

$$\begin{cases} \lambda\sinh\sqrt{\lambda} + \sqrt{\lambda}\cosh\sqrt{\lambda} = -\sqrt{\lambda}\cosh\sqrt{\lambda} - \sinh\sqrt{\lambda} \\ 1 = -2 \quad \text{no solution} \\ +\lambda\sin\sqrt{-\lambda} + \sqrt{-\lambda}\cos\sqrt{-\lambda} = -\sqrt{-\lambda}\cos\sqrt{-\lambda} - \sin\sqrt{-\lambda} \end{cases}$$

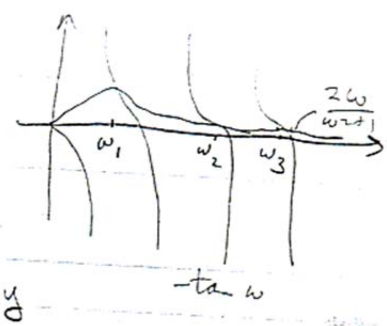
$\lambda>0$: $\dfrac{2\sqrt{\lambda}}{\lambda+1} = -\tanh\sqrt{\lambda}$ or $\dfrac{2\omega}{\omega^2+1} = -\tanh\omega$, if $\lambda=\omega^2$



no solutions.

$\lambda<0$; Take $\lambda=-\omega^2$

$$\frac{2\omega}{\omega^2+1} = -\tan\omega$$



$X_n(x) = \omega_n\cos\omega_n x + \sin\omega_n x$

$Y_n(y) = d_1\cosh\omega_n y + d_2\sinh\omega_n y$

$Y'(0) = Y(0) \implies \omega_n d_2 = d_1$

$Y_n(y) = \omega_n\cosh\omega_n y + \sinh\omega_n y$

(§6.3 #~~18~~ 9)

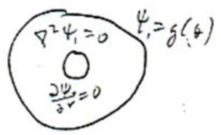
General solⁿ $\psi(x,y) = \sum_{n=1}^{\infty} a_n (\omega_n \cos \omega_n x + \sin \omega_n x) \cdot (\omega_n \cosh \omega_n y + \sinh \omega_n y)$

Bndry cond

$1 = \psi(x,1) = \sum_{n=1}^{\infty} a_n (\omega_n \cos \omega_n x + \sin \omega_n x)(\omega_n \cosh \omega_n + \sinh \omega_n)$

$$\Rightarrow a_n = \frac{\int_0^1 1 (\omega_n \cos \omega_n x + \sin \omega_n x)\, dx}{\int_0^1 (\omega_n \cos \omega_n x + \sin \omega_n x)^2\, dx\, (\omega_n \cosh \omega_n + \sinh \omega_n)}$$

~~#16~~ Subproblem 1:



Ⓗ eigenfunctions are $\cos n\theta$, $\sin n\theta$ (Eq. 33)

$$\left. \begin{array}{ll} R_n(r) = c_1 r^n + c_2 r^{-n} & (n>0), \\ c_1 + c_2 \log r & (n=0) \end{array} \right\} \text{(Eq. 36)}$$

$R'(1) = 0 \Rightarrow n c_1 - n c_2 = 0 \Rightarrow c_1 = c_2 \quad (n>0)$

$\frac{c_2}{1} = 0 \Rightarrow c_2 = 0 \quad (n=0)$

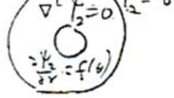
General solution $\psi(r,\theta) = a_0 + \sum_{n=1}^{\infty} a_n \cos n\theta\, (r^n + r^{-n}) + \sum_{n=1}^{\infty} b_n \sin n\theta\, (r^n + r^{-n})$

Bndry. cond $\psi(2,\theta) = g(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)(2^n + 2^{-n})$

$a_0 = \frac{1}{2\pi} \int_0^{2\pi} g(\theta)\, d\theta, \quad a_n = \frac{\int_0^{2\pi} g(\theta) \cos n\theta\, d\theta}{\pi(2^n + 2^{-n})}, \quad b_n = \frac{\int_0^{2\pi} g(\theta) \sin n\theta\, d\theta}{\pi(2^n + 2^{-n})}$

(§6.3 #10)

Subproblem 2:



$$\Theta_n(\theta) = \cos n\theta,\ \sin n\theta \quad , \quad R_n(r) = \begin{cases} c_1 r^n + c_2 r^{-n} & (n>0) \\ c_1 + c_2 \log r & (n=0) \end{cases}$$

$$R_n(2) = 0 \Rightarrow \begin{cases} c_1 2^n + c_2 2^{-n} = 0 & \text{take } c_1 = 2^{-n},\ c_2 = -2^n \\ c_1 + c_2 \log 2 = 0 & \text{take } c_2 = 1,\ c_1 = -\log 2 \end{cases}$$

$$R_n(r) = \begin{cases} \left(\frac{r}{2}\right)^n - \left(\frac{2}{r}\right)^n & n>0 \\ \log\left(\frac{r}{2}\right) & n=0 \end{cases}$$

General soln: $\psi_2(r,\theta) = a_0 \log \frac{r}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \cdot \left[\left(\frac{r}{2}\right)^n - \left(\frac{2}{r}\right)^n\right]$

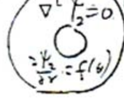
Bndry. cond. $\frac{\partial \psi_2}{\partial r}(1,\theta) = f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \cdot \left[\frac{1}{2^n} - 2^n\right]$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\, d\theta, \quad a_n = \frac{\int_0^{2\pi} f(\theta) \cos n\theta \, d\theta}{\pi\left[\frac{1}{2^n} - 2^n\right]}, \quad b_n = \frac{\int_0^{2\pi} f(\theta) \sin n\theta \, d\theta}{\pi\left[\frac{1}{2^n} - 2^n\right]}$$

Overall $\psi = \psi_1 + \psi_2$.

(§6.3 #10)

Subproblem 2:



$$\Theta_n(\theta) = \cos n\theta,\ \sin n\theta \quad , \quad R_n(r) = \begin{cases} c_1 r^n + c_2 r^{-n} & (n>0) \\ c_1 + c_2 \log r & (n=0) \end{cases}$$

$$R_n(2) = 0 \Rightarrow \begin{cases} c_1 2^n + c_2 2^{-n} = 0 & \text{take } c_1 = 2^{-n},\ c_2 = -2^n \\ c_1 + c_2 \log 2 = 0 & \text{take } c_2 = 1,\ c_1 = -\log 2 \end{cases}$$

$$R_n(r) = \begin{cases} \left(\frac{r}{2}\right)^n - \left(\frac{2}{r}\right)^n & n>0 \\ \log\left(\frac{r}{2}\right) & n=0 \end{cases}$$

General soln: $\psi_2(r,\theta) = a_0 \log\frac{r}{2} + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \cdot \left[\left(\frac{r}{2}\right)^n - \left(\frac{2}{r}\right)^n\right]$

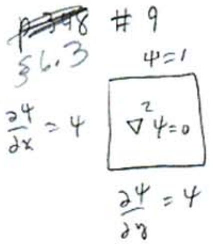
Bndry. cond. $\frac{\partial \psi_2}{\partial r}(1,\theta) = f(\theta) = a_0 + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta) \cdot \left[\frac{1}{2^n} - 2^n\right]$

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)\,d\theta, \quad a_n = \frac{\int_0^{2\pi} f(\theta)\cos n\theta\, d\theta}{\pi\left[\frac{1}{2^n} - 2^n\right]}, \quad b_n = \frac{\int_0^{2\pi} f(\theta)\sin n\theta\, d\theta}{\pi\left[\frac{1}{2^n} - 2^n\right]}$$

Overall $\psi = \psi_1 + \psi_2$.

~~p348~~ # 9 (another solution)

§6.3



$$X(x) = \begin{cases} c_1 \cosh\sqrt{\lambda}\,x + c_2 \sinh\sqrt{\lambda}\,x & \text{if } \lambda > 0 \\ c_1 + c_2 x & \text{if } \lambda = 0 \\ c_1 \cos\sqrt{-\lambda}\,x + c_2 \sin\sqrt{-\lambda}\,x & \text{if } \lambda < 0 \end{cases}$$

Most λ_n's will be negative, so let's explore the last case.

at $x=0$ $\quad \left.\frac{\partial X}{\partial x}\right|_{x=0} = -\sqrt{-\lambda}\, c_1 (0) + \sqrt{-\lambda}\, c_2 (1) = X(0) = c_1$.

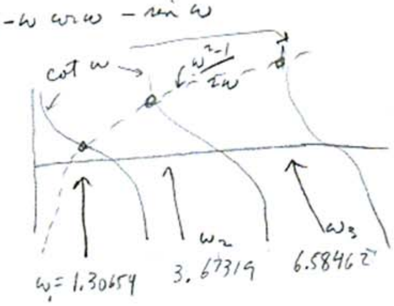
and we take $X(x) = \sqrt{-\lambda}\cos\sqrt{-\lambda}\,x + \sin\sqrt{-\lambda}\,x = \omega\cos\omega x + \sin\omega x$

with $\omega = \sqrt{-\lambda}$, $\lambda = -\omega^2$.

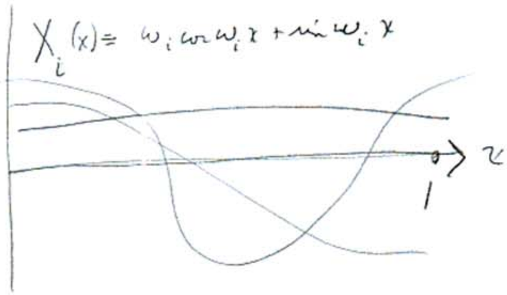
at $x=1$ $\quad \frac{\partial X}{\partial x} = -\omega^2\sin\omega + \omega\cos\omega = -X = -\omega\cos\omega - \sin\omega$

or $(1-\omega^2)\sin\omega = -2\omega\cos\omega$

or $\frac{\omega^2-1}{2\omega} = \cot\omega$.



$\lambda = -1.70705 \quad -13.49232 \quad -43.3572$



$X_1(x)$ has no crossings, so we do not need to consider $\lambda = 0$ or $\lambda > 0$.

$Y(y) = c_1\cosh\omega_i y + c_2 \sinh\omega_i y$

at $y=0$, $Y'(y) = c_1\omega_i \sinh 0 + c_2\omega_i (1) = Y(y) = c_1$

So $Y(y) = \omega_i\cosh\omega_i y + \sinh\omega_i y$

$$\Psi(x,y) = \sum_{n=1}^{\infty} a_n [\omega_n\cos\omega_n x + \sin\omega_n x][\omega_n\cosh\omega_n y + \sinh\omega_n y]$$

$$1 = \sum a_n [\omega_n\cos\omega_n x + \sin\omega_n x][\omega_n\cosh\omega_n + \sinh\omega_n]$$

$$a_n = \int_0^1 (1)[\omega_n\cos\omega_n x + \sin\omega_n x]\,dx \Big/ \int_0^1 [\omega_n\cos\omega_n x + \sin\omega_n x]^2 dx\, [\omega_n\cosh\omega_n + \sinh\omega_n]$$

p. 348 #10. §6.3

The Θ eigenfunctions are the Fourier series.

$$\Theta'' = \lambda\Theta \qquad \lambda = -n^2$$

From Example 2, p. 297, with $K=0$ & $\lambda=-n^2$, the radial equation is equidimensional;

$$r^2 R'' + r R' - n^2 R = 0$$

and its solutions are r^n and r^{-n} for $n>0$, $a_1 r^n + a_2 r^{-n}$

and 1 and $\ln r$ for $n=0$. $a_1 + a_2 \ln r$

at $r=1$, $R(r)=0$:

$$a_1(1)^n + a_2(1)^{-n} = 0 \implies a_2 = -a_1, \quad R = r^n - r^{-n}$$

$$a_1 + a_2 \ln 1 = 0 \implies a_1 = 0$$

General soln:

$$\psi(r,\theta) = c_0 \ln r + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta)(r^n - r^{-n})$$

at $r=2$, $\psi(2,\theta) = g(\theta) = c_0 \ln 2 + \sum_{n=1}^{\infty} (c_n \cos n\theta + d_n \sin n\theta)(2^n - 2^{-n})$

$$\therefore \quad c_0 = \frac{\int_0^{2\pi} g(\theta)\,d\theta}{\int_0^{2\pi} 1^2\,d\theta\,(\ln 2)} = \frac{\int_0^{2\pi} g(\theta)\,d\theta}{2\pi \ln 2}$$

$$\text{for } n>0 \quad c_n = \frac{\int_0^{2\pi} g(\theta)\cos n\theta\,d\theta}{\int_0^{2\pi} \cos^2 n\theta\,d\theta\,(2^n - 2^{-n})} = \frac{\int_0^{2\pi} g(\theta)\cos n\theta\,d\theta}{\pi(2^n - 2^{-n})}$$

$d_n =$ ' ' '

§6.4 #2

$$\psi_1=0 \;\boxed{\begin{array}{c}\psi_1=0\\ \nabla^2\psi_1=0\\ \psi_1=0\end{array}}\; \psi_1=g(y) \qquad \psi_2=\sin y \;\boxed{\begin{array}{c}\psi_2=0\\ \nabla^2\psi_2=0\\ \psi_2=0\end{array}}\; \psi_2=0$$

Subproblem 1. $\psi_1(x,y)=\sum_{n=1}^{\infty} a_n \sin ny \sinh nx$ as usual.

Bndry cond. $\psi(L,y)=g(y)=\sum_{n=1}^{\infty} a_n \sin ny \sinh nL$

$$a_n=\frac{\int_0^{\pi} g(y)\sin ny\,dy}{\underbrace{\int_0^{\pi}\sin^2 ny\,dy}_{\pi/2}\,\sinh nL}$$

$$\psi_1(x,y)=\frac{2}{\pi}\sum_{n=1}^{\infty}\sin ny\int_0^{\pi} g(y)\sin ny\,dy\;\frac{\sinh nx}{\sinh nL} \to \text{approaches } 0 \text{ as } L\to\infty$$

in text, $\psi_2(x,y)=\sin y\,[\cosh x-\coth L\sinh x]\to e^{-x}\sin y$

$\therefore\ \psi_1+\psi_2\to e^{-x}\sin y$ as $L\to\infty$.

#1 As usual $Y_n(y)=\sin ny$, $\cancel{X_n(x)=c_1\cosh nx+c_2\sinh nx}$

measure x relative to $x=L$; $X_n(x)=c_1\cosh n(L-x)+c_2\sinh n(L-x)$

$X'(L)=-3X(L)\Rightarrow -c_2 n=-3c_1 \Rightarrow c_2=\frac{3}{n}c_1$

So $X_n(x)=\cosh n(L-x)+\frac{3}{n}\sinh n(L-x)$

$\psi(x,y)=\sum_{n=1}^{\infty} a_n\sin ny\,\left[\cosh n(L-x)+\frac{3}{n}\sinh n(L-x)\right]$

Bndry. cond. at $x=0$: $\sin y=\sum_{n=1}^{\infty} a_n\sin ny\,\left[\cosh nL+\frac{3}{n}\sinh nL\right]$

Clearly $a_{n>1}=0$, $a_1=\frac{1}{\cosh L+3\sinh L}$

$$\psi(x,y)=\sin y\,\frac{\cosh(L-x)+3\sinh(L-x)}{\cosh L+3\sinh L}$$

$$=\sin y\,\frac{\frac{e^{L-x}+e^{-L+x}}{2}+3\,\frac{e^{L-x}-e^{-L+x}}{2}}{\frac{e^{L}+e^{-L}}{2}+3\,\frac{e^{L}-e^{-L}}{2}} \underset{\text{for } L\to\infty}{\simeq} \frac{2e^{L-x}}{2e^{L}}\sin y\to e^{-x}\sin y$$

§6.4 #4 $\Psi(x,y) = X(x)\,Y(y)$

$$X(x) = \begin{cases} c_1 \cosh\sqrt{\lambda}\, x + c_2 \sinh\sqrt{\lambda}\, x \\ c_1 + c_2 x \\ c_1 \cos\sqrt{-\lambda}\, x + c_2 \sin\sqrt{-\lambda}\, x \end{cases}$$

$X'(0) = 0 \Rightarrow c_2 = 0.$

$X(\infty)$ finite $\Rightarrow$ $X = \cos\sqrt{-\lambda}\, x$, cosine transform,

So take $X(x) = \cos\omega x$, $\lambda = -\omega^2$

$Y(y) = c_1 \cosh\omega y + c_2 \sinh\omega y$. $\quad Y(0) = 0 \Rightarrow c_1 = 0$

$$\Psi(x,y) = \int_0^\infty C(\omega) \cos\omega x \sinh\omega y \, d\omega$$

Bndry cond. $\frac{\partial\Psi}{\partial y}(x,\pi) = g(x) = \int_0^\infty C(\omega) \cos\omega x \;\omega\cosh\omega\pi \, d\omega$

So $C(\omega)\,\omega\cosh\omega\pi$ is the cosine transform of $g(x)$. Equivalently, from Table 6.3 #9,

$$\int_0^\infty g(x) \cos\omega' x \, dx = \int_0^\infty\!\!\int_0^\infty C(\omega) \cos\omega x \cos\omega' x \;\omega\cosh\omega\pi \, d\omega\, dx$$

$$= \frac{\pi}{2} \int_0^\infty C(\omega)\, \omega\cosh\omega\pi \;\delta(\omega-\omega')\, d\omega$$

$$= \frac{\pi}{2}\, C(\omega')\, \omega' \cosh\omega'\pi$$

Therefore

$$C(\omega) = \frac{2}{\pi\,\omega\cosh\omega\pi} \int_0^\infty g(x) \cos\omega x \, dx \quad \text{(dropping the "prime")}$$

§6.4
~~#6~~
#5

$\psi_1 = g(x)$; $\nabla^2 \psi_1 = 0$; $\psi_1 = 0$

$\psi_2 = 0$; $\nabla^2 \psi_2 = 0$; $\psi_2 = h(x)$

Subproblem 1. $\psi_1(x,y) = X(x)\,Y(y)$

$X'' = \lambda X$, X finite at $\pm\infty$

$\Rightarrow$ Fourier transform, $X = e^{i\omega x}$; $\lambda = -\omega^2$

$Y_\omega(y) = c_1 \cosh \omega y + c_2 \sinh \omega y \qquad Y_\omega(0) = 0 = c_1$

$$\psi_1(x,y) = \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x} \sinh \omega y \, d\omega$$

Bndry. cond. $\psi_1(x,\pi) = g(x) = \int_{-\infty}^{\infty} F(\omega)\, e^{i\omega x} \sinh \omega\pi \, d\omega$

Table 6.3 #7 $\int_{-\infty}^{\infty} g(x)\, e^{-i\omega' x} dx = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} F(\omega) e^{i\omega x} e^{-i\omega' x} \sinh \omega\pi \, dx\, d\omega$

$$= \int_{-\infty}^{\infty} F(\omega)\, 2\pi\,\delta(\omega-\omega') \sinh \omega\pi \, d\omega$$

$$= 2\pi F(\omega') \sinh \omega'\pi$$

Therefore $F(\omega) = \frac{1}{2\pi \sinh \omega\pi} \int_{-\infty}^{\infty} g(x)\, e^{-i\omega x} dx.$

Subproblem 2. $X_\omega(x)$ as before; measure y from π.

$Y_\omega(y) = c_1 \cosh \omega(\pi - y) + c_2 \sinh \omega(\pi - y) \qquad Y_\omega(\pi) = c_1 = 0$

$$\psi_2(x,y) = \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega x} \sinh \omega(\pi - y)\, d\omega$$

at $y = 0$

$$\psi_2(x,0) = h(x) = \int_{-\infty}^{\infty} G(\omega)\, e^{i\omega x} \sinh \omega\pi \, d\omega$$

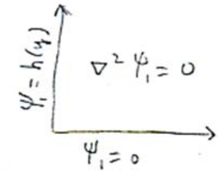
solve as above:

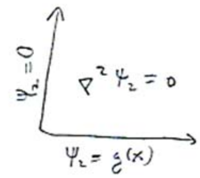
$$G(\omega) = \frac{1}{2\pi \sinh \omega\pi} \int_{-\infty}^{\infty} h(x)\, e^{-i\omega x}\, dx.$$

$\psi = \psi_1 + \psi_2$

6

§6.4 # [illegible]





Refer to Example 5, subproblem 2.

Subproblem 1. $r^2 R'' + rR' = -\lambda R$ $\quad R(0)$ finite, $R(\infty)$ finite

Table 6.3 #15 $(\sigma = 1,\ "\lambda" = -\lambda)$

$$R_\omega(r) = r^{-i\omega} = e^{-i\omega \log r} \qquad \text{weight } \rho(r) = r^{-1}$$

$$"\lambda" = -\omega^2, \quad -\infty < \omega < \infty \quad (\text{so } \underline{\text{our}}\ \lambda = +\omega^2).$$

misprints ~~[illegible]~~ ~~[illegible]~~ it runs from $\underline{0}$ to ∞. (but ω still runs from $-\infty$ to ∞)

$$\Theta_\omega'' = \lambda \Theta_\omega = \omega^2 \Theta_\omega, \quad \text{so } \Theta_\omega(\theta) = C_1 \cosh \omega\theta + C_2 \sinh \omega\theta$$

$$\Theta_\omega(0) = 0 \Rightarrow C_1 = 0$$

$$\psi_1(r,\theta) = \int_{-\infty}^{\infty} G(\omega)\, r^{-i\omega} \sinh \omega\theta \, d\omega$$

note: $y = r$ when $\theta = \pi/2$.

Bndry cond.; $\psi_1(r, \tfrac{\pi}{2}) = h(r) = \int_{-\infty}^{\infty} G(\omega) r^{-i\omega} \sinh \omega\tfrac{\pi}{2}\, d\omega$

From the Table,
$$\int_0^\infty r^{-i\omega}\, r^{+i\omega'}\, r^{-1}\, dr = 2\pi\,\delta(\omega - \omega')$$

$$\text{So } \int_0^\infty h(r)\, r^{i\omega'} r^{-1} dr = \int_0^\infty\!\!\int_{-\infty}^{\infty} r^{-i\omega} r^{i\omega'} G(\omega) \sinh \omega\tfrac{\pi}{2}\, r^{-1}\, d\omega\, dr$$

$$= \int_{-\infty}^{\infty} G(\omega) \sinh \frac{\omega\pi}{2}\, 2\pi\, \delta(\omega - \omega')\, d\omega$$

$$= 2\pi\, G(\omega') \sinh \frac{\omega'\pi}{2}$$

$$\text{Therefore } G(\omega) = \frac{1}{2\pi \sinh \omega\pi/2} \int_0^\infty h(r)\, r^{i\omega - 1}\, dr$$

Subproblem 2. Same as subproblem 1 with θ measured from $\pi/2$,

$$\psi_2(r,\theta) = \int_{-\infty}^{\infty} H(\omega)\, r^{-i\omega} \sinh \omega\left(\tfrac{\pi}{2} - \theta\right) d\omega$$

$$H(\omega) = \frac{1}{2\pi \sinh \omega\pi/2} \int_0^\infty g(r)\, r^{i\omega - 1}\, dr$$

$$\psi = \psi_1 + \psi_2$$

§7.1 #1 To meet the boundary condition involving f_4 we simply switch the roles of "x" and "z" in the solution for f_2;

$$\psi_4 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} b_{mn} \sin \frac{m\pi z}{c} \cos \frac{n\pi y}{b} \sinh \sqrt{\frac{m^2\pi^2}{c^2} + \frac{n^2\pi^2}{b^2}}\, x,$$

$$b_{pq} = \frac{\int_0^c \int_0^b f_4(y,z) \sin \frac{p\pi z}{c} \cos \frac{q\pi y}{b}\, dy\, dz}{\int_0^c \sin^2 \frac{p\pi z}{c}\, dz \int_0^b \cos^2 \frac{q\pi y}{b}\, dy \sinh \sqrt{\frac{p^2\pi^2}{c^2} + \frac{q^2\pi^2}{b^2}}\, a}$$

To meet the boundary condition involving f_1 we measure z from the top $(z=c)$ and substitute into the solution for f_2:

$$\psi_1 = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} c_{m,n} \sin \frac{m\pi x}{a} \cos \frac{n\pi y}{b} \sinh \sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{b^2}}\, (c-z),$$

$$c_{pq} = \frac{\int_0^a \int_0^b f_1(x,y) \sin \frac{p\pi x}{a} \cos \frac{q\pi y}{b}\, dy\, dx}{\int_0^a \sin^2 \frac{p\pi x}{a}\, dx \int_0^b \cos^2 \frac{q\pi y}{b}\, dy \sinh \sqrt{\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{b^2}}\, c}$$

To meet the boundary condition involving f_3 we measure x from the right face $(x=a)$ and substitute into the solution for f_4:

$$\psi_3(x,y,z) = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} d_{m,n} \sin \frac{m\pi z}{c} \cos \frac{n\pi y}{b} \sinh \sqrt{\frac{m^2\pi^2}{c^2} + \frac{n^2\pi^2}{b^2}}\, (a-x),$$

$$d_{p,q} = \frac{\int_0^c \int_0^b f_3(y,z) \sin \frac{p\pi z}{c} \cos \frac{q\pi y}{b}\, dy\, dz}{\int_0^c \sin^2 \frac{p\pi z}{c}\, dz \int_0^b \cos^2 \frac{q\pi y}{b}\, dy \sinh \sqrt{\frac{p^2\pi^2}{c^2} + \frac{q^2\pi^2}{b^2}}\, a}$$

(§7.1 #1) To meet the boundary condition involving f_6, we need to find X and Z eigenfunctions. Clearly $X_m(x) = \sin\frac{m\pi x}{a}$, $m = 1, 2, \dots$; $Z_n(z) = \sin\frac{n\pi z}{c}$, $n = 1, 2, \dots$. Therefore

$$Y'' = \sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{c^2}}\; Y, \quad Y = c_1 \cosh\sqrt{\cdots}\, y + c_2 \sinh\sqrt{\cdots}\, y.$$

The solution satisfying $Y'(0) = 0$ is $\cosh\sqrt{\cdots}\, y$.

$$\psi_6(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} e_{m,n} \sin\frac{m\pi x}{a} \sin\frac{n\pi z}{c} \cosh\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{c^2}}\; y,$$

Evaluating $\frac{\partial \psi_6}{\partial y}$,

$$e_{pq} = \frac{\int_0^a \int_0^c f_6(x,z) \sin\frac{p\pi x}{a} \sin\frac{q\pi z}{c}\, dz\, dx}{\int_0^a \sin^2\frac{p\pi x}{a}\, dx \int_0^c \sin^2\frac{q\pi z}{c}\, dz \sqrt{\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{c^2}} \sinh\sqrt{\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{c^2}}\; b}$$

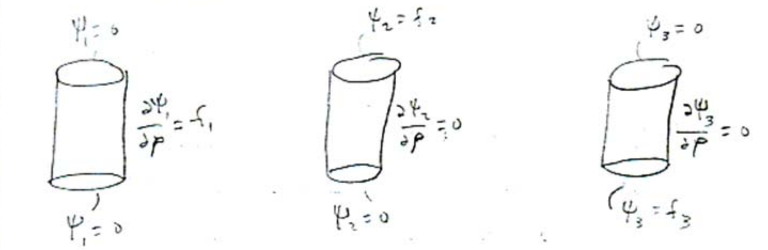
For the boundary condition with f_5, measure y from the back ($y = b$) in the preceding:

$$\psi_5(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} f_{mn} \sin\frac{m\pi x}{a} \sin\frac{n\pi z}{c} \cosh\sqrt{\frac{m^2\pi^2}{a^2} + \frac{n^2\pi^2}{c^2}}\; (b-y)$$

Note that a $(-)$ sign comes in when we take $\frac{\partial \psi_5}{\partial y}$:

$$f_{pq} = -\frac{\langle f_5(x,z), \sin\frac{p\pi x}{a} \sin\frac{q\pi z}{c} \rangle}{\| \sin\frac{p\pi x}{a}\|^2\, \|\sin\frac{q\pi z}{c}\|^2 \sqrt{\frac{p^2\pi^2}{a^2} + \frac{q^2\pi^2}{b^2}} \sinh\sqrt{\frac{p^2\pi^2}{a^2} - \frac{q^2\pi^2}{b^2}}\; b}$$

§7.1 #3



Subproblem 1 has the same general solution as example 2, subproblem 1, to

$$\psi_1(r,\theta,z) = \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} a_{nm} \cos n\theta \sin \frac{m\pi z}{c} I_n\left(\frac{m\pi}{c}\rho\right)$$
$$+ \sum_{1}^{\infty} \sum_{1}^{\infty} b_{mn} \sin n\theta \left(\;''\; \right)\left(\;''\; \right).$$

Bndry. cond.

$$\frac{\partial \psi_1}{\partial r}(a,\theta,z) = f_1(\theta,z) = \sum_0^{\infty}\sum_1^{\infty}(\ldots\ldots) \frac{m\pi}{c} I_n'\left(\frac{m\pi a}{c}\right)$$
$$+ \sum_1^{\infty}\sum_1^{\infty}(\ldots\ldots) \frac{m\pi}{c} I_n'\left(\frac{m\pi a}{c}\right)$$

Therefore

$$a_{nm} = \frac{\left\langle f_1(\theta,z), \cos n\theta \sin \frac{m\pi z}{c} \right\rangle}{\|\cos n\theta\|^2 \, \|\sin \frac{m\pi z}{c}\|^2 \, \frac{m\pi}{c} I_n'\left(\frac{m\pi a}{c}\right)}$$

$$b_{nm} = \frac{\left\langle f_1(\theta,z), \sin n\theta \sin \frac{m\pi z}{c} \right\rangle}{\|\sin n\theta\|^2 \, \|\sin \frac{m\pi z}{c}\|^2 \, \frac{m\pi}{c} I_n'\left(\frac{m\pi a}{c}\right)}$$

Subproblem 2. As before, $\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$

The R-equations are $\rho^2 R'' + \rho R' - n^2 R = -\mu\rho^2 R$, $R(0)$ finite, $R'(a) = 0$

Table 6.3 # 17: $(\nu = n, \mu = -\lambda, a = b)$

$$R_{n,p}(\rho) = J_n(j'_{n,p}\rho/a), \quad \mu_{n,p} = (j'_{n,p}/a)^2 \quad p = 1, 2, \ldots$$

also $R_{0,0}(\rho) = 1$, $\mu_{0,0} = 0$

$Z'' = \mu Z \Rightarrow Z_{n,p} = c_1 \cosh \frac{j'_{np} z}{a} + c_2 \sinh \frac{j'_{np} z}{a}$ & $c_1 = 0$

except $Z_{0,0} = c_1 + c_2 z$ & $c_1 = 0$ because $Z'(0) = 0$

(§7.7 #3) General solution

$$\Psi_z(r,\theta,z) = a_{00}\, z + \sum_{p=1}^{\infty} a_{0p}\, J_0(j'_{0,p}\rho/a)\sinh\frac{j'_{0,p}z}{a}$$

$$+\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} a_{np}\cos n\theta\, J_n(j'_{n,p}\rho/a)\sinh \frac{j'_{n,p}z}{a}$$

$$+\sum_{n=1}^{\infty}\sum_{p=1}^{\infty} b_{np}\sin n\theta\, J_n(j'_{n,p}\rho/a)\sinh \frac{j'_{n,p}z}{a}$$

Bndry. cond. $\Psi_z(\rho,\theta,c) = f_z(\rho,\theta)$

$$f_z(\rho,\theta) = a_{00}\,c + \sum_{p=1} a_{0p} J_0\left(\frac{j'_{0,p}\rho}{a}\right)\sinh\frac{j'_{0,p}c}{a} + \sum_{n\neq 1}^{\infty}\sum_{p}^{\infty}\left(a_{np}\cos n\theta + b_{np}\sin n\theta\right) \cdot J_n\left(\frac{j'_{n,p}\rho}{a}\right)\sinh\frac{j'_{n,p}c}{a}$$

~~For multiply both sides by $\cos n'\theta$ and integrate~~: $\int_0^{2\pi}\cos n\theta \sin n'\theta\, d\theta = 0$

$\int_0^{2\pi}\sin n\theta \sin n'\theta\, d\theta = \pi\,\delta_{nn'}$

~~$\int_0^{2\pi} f_z(\rho,\theta)\cos n\theta\, d\theta =$~~

From the orthogonality of the Fourier series, the coefficient of $\cos n\theta$ is given by $\int_0^{2\pi} f_z(\rho,\theta)\cos n\theta\, d\theta$ times $\frac{1}{\pi}$ ($\frac{1}{2\pi}$ for $n=0$): so

$$a_{00}c + \sum_{p=1}^{\infty} a_{0p} J_0\left(\frac{j'_{0,p}\rho}{a}\right)\sinh\frac{j'_{0,p}c}{a} = \frac{1}{2\pi}\int_0^{2\pi} f_z(\rho,\theta)\,d\theta$$

$$\sum_{p=1}^{\infty} a_{n,p} J_n\left(\frac{j'_{n,p}\rho}{a}\right)\sinh\frac{j'_{n,p}c}{a} = \frac{1}{\pi}\int_0^{2\pi} f_z(\rho,\theta)\cos n\theta\, d\theta$$

and similarly

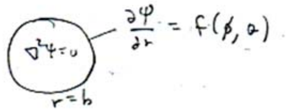
$$\sum_{p=1}^{\infty} b_{np} J_n\left(\frac{j'_{n,p}\rho}{a}\right)\sinh\frac{j'_{n,p}c}{a} = \frac{1}{\pi}\int_0^{2\pi} f_z(\rho,\theta)\sin n\theta\, d\theta$$

From the orthogonality integrals in Table 6.3, #17, then,

(§7.1 #3)

Subproblem 3. Replace "z" by $c-z$ and "f_2" with f_3 in subproblem 2.

~~§7.2 #8~~ Dropped from text (?)



Note: for the equilibrium (Laplace) equation to hold, the net influx of heat must be zero. So $0 = \iint_{\text{surface}} f(\phi,\theta)\, d(\text{area}) = b^2 \int_0^{2\pi}\int_0^{\pi} f(\phi,\theta) \sin\phi \, d\phi \, d\theta$

Since Θ is periodic and Φ is finite at the poles, $\psi = R(r)\, Y_l^m(\theta,\phi)$.
Eq. for R is same, $R(0)$ finite, so the general solution is the same as Example 4:

$$\psi(r,\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}\, r^l\, Y_l^m(\theta,\phi)$$

At $r=b$, $\frac{\partial\psi}{\partial r} = f(\phi,\theta) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm}\, l\, b^{l-1}\, Y_l^m(\theta,\phi)$.

So $\int_0^{2\pi}\int_0^{\pi} f(\phi,\theta)\, \overline{Y_{l'}^{m'}(\phi,\theta)} \sin\phi \, d\phi\, d\theta$

$$= \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm}\, l\, b^{l-1} \underbrace{\int_0^{2\pi}\int_0^{\pi} Y_l^m(\theta,\phi)\overline{Y_{l'}^{m'}(\theta,\phi)} \sin\phi\, d\phi\, d\theta}_{\delta_{ll'}\,\delta_{mm'}}$$

$$= a_{l'm'}\, l'\, b^{l'-1}$$

So for $l>0$

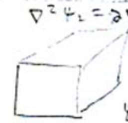
$$a_{lm} = \frac{1}{l b^{l-1}} \int_0^{2\pi}\int_0^{\pi} f(\phi,\theta)\, \overline{Y_l^m(\phi,\theta)} \sin\phi\, d\phi\, d\theta .$$

For $l=0$ the previous eq. says $\int_0^{2\pi}\int_0^{\pi} f\, \overline{Y_l^m} \cdots = a_{00}\cdot 0$. But that's consistent, by the steady-state (Neumann) condition. a_{00} is undeterminable from the problem data: ψ is only determined up to a constant.

$\nabla^2\psi_1 = 0$

$\psi_1 = 1$

$\nabla^2\psi_2 = \partial\psi_2/\partial t$



$\psi_2 = 0$ on all sides

$\psi_2(x,y,z,0) = f - \psi_1$

§7.2 #2.

Subproblem 1. Y + Z satisfy homogeneous Dirichlet conditions

$Y_m(y) = \sin\frac{m\pi y}{b} \quad Z_n(z) = \sin\frac{n\pi z}{c}$

$X_{mn}(x) = \sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,(a-x)$ so $X_{mn}(a) = 0$.

$$\psi_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,(a-x),$$

$$a_{mn} = \frac{\int_0^b\int_0^c \sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\,dz\,dy}{\|\sin\frac{m\pi y}{b}\|^2\,\|\sin\frac{n\pi z}{c}\|^2\sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,a}$$

Subproblem 2.

X, Y, + Z are all eigenfunctions.

$X_l = \sin\frac{l\pi x}{a} \quad Y_m = \sin\frac{m\pi y}{b} \quad Z_n = \sin\frac{n\pi z}{c}$

$T_{lmn}(t) = \exp\left[-\sqrt{\frac{l^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\;t\right]$

General soln: $\psi_2(x,y,z,t) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{lmn}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\,e^{-\sqrt{\cdots}\,t}$

Initial condition: $\psi_2(x,y,z,0) = f(x,y,z) - \psi_1(x,y,z)$

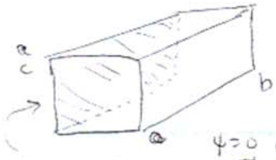
Therefore

$$a_{lmn} = \frac{\int_0^a\int_0^b\int_0^c [f(x,y,z)-\psi_1(x,y,z)]\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}}{\|\sin\frac{l\pi x}{a}\|^2\,\|\sin\frac{m\pi y}{b}\|^2\,\|\sin\frac{n\pi z}{c}\|^2}$$

$\psi = \psi_1 + \psi_2$

§1,2

$\boxed{\S\ \#2}$



$\psi(0,y,z,t)=1$ $\quad$ $\psi = 0$ on other 5 faces

$$\frac{\partial \psi}{\partial t} = \nabla^2 \psi \qquad \psi(x,y,z,0) = g(x,y,z)$$

Steady-state: $\nabla^2 \psi_\infty = 0$ $\quad$ $\psi_\infty = 0$ on all faces except $\psi_\infty(0,y,z)=1$

$$Z'' = \lambda Z \Rightarrow Z(z) = \sin\frac{n\pi z}{c}$$

$$Y'' = \mu Y \Rightarrow Y(y) = \sin\frac{m\pi y}{b}$$

$$X'' = (-\lambda-\mu)X \quad X(x) = \sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,(a-x)$$

$$\psi_\infty(x,y,z) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} a_{mn} \sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,(a-x)\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}$$

at $x=0$ $\quad 1 = \sum\sum a_{mn}\sinh\sqrt{\cdots}\,a \sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}$

$$a_{mn} = \frac{\langle 1, \sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\rangle}{\sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,a\,\|\sin\frac{m\pi y}{b}\|^2\,\|\sin\frac{n\pi z}{c}\|^2}$$

$$= \frac{\int_0^b\int_0^c \sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\,dz\,dy}{\sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,a\left(\frac{b}{2}\right)\left(\frac{c}{2}\right)}$$

$$= \frac{4\,\frac{bc}{mn\pi^2}[(-1)^m-1][(-1)^n-1]}{bc\sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,a} = \frac{4[(-1)^m-1][(-1)^n-1]}{mn\pi^2\sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,a}$$

So $$\psi_\infty(x,y,z) = \frac{4}{\pi^2}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{[(-1)^m-1][(-1)^n-1]}{mn\sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,a}\sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,(a-x)\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}$$

Transient: $\frac{\partial \psi_0}{\partial t} = \nabla^2\psi_0$, $\quad \psi_0(x,y,z,0) = g(x,y,z) - \psi_\infty(x,y,z)$ $\quad \psi_0 = 0$ on all sides

$$Z(z) = \sin\frac{n\pi z}{c} \quad Y(y) = \sin\frac{m\pi y}{b} \quad X(x) = \sin\frac{l\pi x}{a}$$

$$T' = \left(-\frac{l^2\pi^2}{a^2}-\frac{m^2\pi^2}{b^2}-\frac{n^2\pi^2}{c^2}\right)T \qquad T(t) = e^{-\left(\frac{l^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}\right)t}$$

$$\psi_0(x,y,z,t) = \sum_{l=1}^{\infty}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} b_{lmn}\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\cdot \exp\left(-\left[\frac{l^2\pi^2}{a^2}+\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}\right]t\right).$$

(§7.2 #2 cont.)

$$b_{lmn} = \frac{\langle g - \psi_\infty, \sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\rangle}{\|\sin\frac{l\pi x}{a}\|^2\|\sin\frac{m\pi y}{b}\|^2\|\sin\frac{n\pi z}{c}\|^2}$$

$$= \frac{1}{\frac{a}{2}\frac{b}{2}\frac{c}{2}}\int_0^a\int_0^b\int_0^c g(x,y,z)\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\,dz\,dy\,dx$$

$$- \frac{1}{\frac{a}{2}\frac{b}{2}\frac{c}{2}}\frac{4}{\pi^2}\sum_{m'=1}^{\infty}\sum_{n'=1}^{\infty}\frac{[(-1)^{m'}-1][(-1)^{n'}-1]}{m'n'\sinh\sqrt{\frac{m'^2\pi^2}{b^2}+\frac{n'^2\pi^2}{c^2}}\,a}\int_0^a \sinh\sqrt{\frac{m'^2\pi^2}{a^2}+\frac{n'^2\pi^2}{b^2}}(a-x)\sin\frac{l\pi x}{a}\,dx.$$

$$\cdot\underbrace{\int_0^b \sin\frac{m'\pi y}{b}\sin\frac{m\pi y}{b}\,dy}_{\frac{b}{2}\delta_{m'm}}\cdot\underbrace{\int_0^c \sin\frac{n'\pi z}{c}\sin\frac{n\pi z}{c}\,dz}_{\frac{c}{2}\delta_{n'n}}$$

$$b_{lmn} = \frac{8}{abc}\int_0^a\int_0^b\int_0^c g(x,y,z)\sin\frac{l\pi x}{a}\sin\frac{m\pi y}{b}\sin\frac{n\pi z}{c}\,dz\,dy\,dx$$

$$- \frac{8}{a\pi^2}\,\frac{[(-1)^m-1][(-1)^n-1]}{mn\sinh\sqrt{\frac{m^2\pi^2}{b^2}+\frac{n^2\pi^2}{c^2}}\,a}\int_0^a \sinh\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}}(a-x)\sin\frac{l\pi x}{a}\,dx.$$

6

§7.2 #8 $\psi_1(x)''=0$, $\psi_1=1$ ——— $\psi_1=2$ $\qquad$ $\frac{\partial\psi_2}{\partial t}=\frac{\partial^2\psi_2}{\partial x^2}$, $\psi_2=0$ ——— $\psi_2=0$

$\psi_2(x,0)=1+x-\psi_1(x)$

Subproblem 1. $\psi_1=a+bx = 1+x$ to fit the boundary conditions.
Subproblem 2 thus has $\psi_2(x,0)=0$ as its initial condition, with homogeneous boundary conditions. It is trivial: $\psi_2=0$.
$\psi(x,t)=1+x$.

§7.2 #9 $\psi_1(x)''=0$, $\psi_1=1$ ——— $\frac{\partial\psi}{\partial x}=0$ $\qquad$ $\frac{\partial\psi_2}{\partial t}=\frac{\partial^2\psi_2}{\partial x^2}$, $\psi_2=0$ ——— $\frac{\partial\psi_2}{\partial x}=0$

$\psi_2(x,0)=f(x)-\psi_1$

Subproblem 1. $\psi_1=a+bx=1$ to fit the boundary conditions.
Subproblem 2. $X(x)=c_1\cos\sqrt{-\lambda}\,x+c_2\sin\sqrt{-\lambda}\,x$ (for $\lambda<0$)*

$X(0)=c_1=0 \quad X'(\tfrac{\pi}{2})=\sqrt{-\lambda}\cos\sqrt{-\lambda}\,\tfrac{\pi}{2}=0$

$\Rightarrow \sqrt{-\lambda}=2n+1 \quad X_n(x)=\sin(2n+1)x \quad n=0,1,2,\ldots$

*(note: since $X_0(x)=\sin x$ has no interior zeros, this must be the first eigenfunction & we don't have to check $\lambda\geq 0$.)

$$\psi_2(x,t)=\sum_{n=0}^{\infty}a_n\sin(2n+1)x\,e^{-(2n+1)^2t}$$

Initial condition:

$$\psi_2(x,0)=f(x)-1=\sum_{n=0}^{\infty}a_n\sin(2n+1)x$$

$$a_n=\frac{\langle f(x)-1,\sin(2n+1)x\rangle}{\|\sin(2n+1)x\|^2}$$

§7.3 #3 $X_m(x) = \sin\frac{m\pi x}{a}$ $Y_n(y) = \sin\frac{n\pi y}{b}$ for Dirichlet conditions.

$$\psi(x,y,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\left[a_{mn}\cos\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}}\,t + b_{mn}\sin\sqrt{\cdots}\,t\right].$$

The eigenfrequencies are $\left\{\sqrt{\frac{m^2\pi^2}{a^2}+\frac{n^2\pi^2}{b^2}},\ m=1,2,\ldots;\ n=1,2,\ldots\right\}$.

§7.3 #6 General solution, from example 6, is

$$\psi(\rho,\theta,z,t) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\int_{k=-\infty}^{\infty} J_n(j_{n,p}\rho/b)\, e^{ikz}\left[a_{np}(k)\cos n\theta + b_{np}(k)\sin n\theta\right]\sin\omega_{npk}t\, dk$$

(ω_{npk} defined by eq. (35).)

$$+ \sum_{p=1}^{\infty}\int_{k=-\infty}^{\infty} J_0(j_{0,p}\rho/b)\, e^{ikz}\, a_{0p}(k)\sin\omega_{0pk}t\, dk$$

Initial condition $\frac{\partial\psi}{\partial t}(\rho,\theta,z,0) = g(\rho,\theta,z)$

$$g(\rho,\theta,z) = \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\int_{k=-\infty}^{\infty} J_n(j_{n,p}\rho/b)\, e^{ikz}\left[a_{np}(k)\cos n\theta + b_{np}(k)\sin n\theta\right]\omega_{npk}\, dk$$

$$+ \sum_{p=1}^{\infty}\int_{k=-\infty}^{\infty} J_0(j_{0,p}\rho/b)\, e^{ikz}\, a_{0p}(k)\,\omega_{0pk}\, dk$$

$$\int_{z=-\infty}^{\infty}\int_{\rho=0}^{b}\int_{\theta=0}^{2\pi} g(\rho,\theta,z)\, e^{-ik'z} J_{n'}(j_{n',p'}\rho/b)\cos n'\theta\, d\theta\, \rho\, d\rho\, dz$$

$$= \sum_{n=1}^{\infty}\sum_{p=1}^{\infty}\int_{k=-\infty}^{\infty}\int_0^b J_n(j_{np}\rho/b)\, J_{n'}(j_{n'p'}\rho/b)\,\rho\, d\rho \cdot$$

$$\cdot\underbrace{\int_0^{2\pi}\cos n\theta\cos n'\theta\, d\theta}_{\pi\delta_{nn'}} \cdot \underbrace{\int_{-\infty}^{\infty} e^{ikz}e^{-ik'z}\, dz}_{2\pi\,\delta(k-k')\ \text{Table 6.3 \#7}}\ a_{np}(k)\,\omega_{npk}\, dk$$

6

§7.2 #8

$$\psi_1=1 \;\overset{\psi_1(x)''=0}{\rule{2cm}{0.4pt}}\; \psi_1=2 \qquad\qquad \psi_2=0 \;\overset{\frac{\partial\psi_2}{\partial t}=\frac{\partial^2\psi_2}{\partial x^2}}{\rule{2cm}{0.4pt}}\; \psi_2=0$$

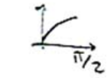
$$\psi_2(x,0) = 1+x-\psi_1(x)$$

Subproblem 1. $\psi_1 = a+bx = 1+x$ to fit the boundary conditions.
Subproblem 2 then has $\psi_2(x,0)=0$ as its initial condition, with homogeneous boundary conditions. It is trivial: $\psi_2=0$.
$\psi(x,t)=1+x$.

§7.2 #9 7

$$\psi_1=1 \;\overset{\psi_1(x)''=0}{\rule{2cm}{0.4pt}}\; \frac{\partial\psi}{\partial x}=0 \qquad\qquad \psi_2=0 \;\overset{\frac{\partial\psi_2}{\partial t}=\frac{\partial^2\psi_2}{\partial x^2}}{\rule{2cm}{0.4pt}}\; \frac{\partial\psi_2}{\partial x}=0$$

$$\psi_2(x,0) = \underbrace{\cancel{\cos x}}_{f(x)} - \psi_1$$

Subproblem 1. $\psi_1 = a+bx = 1$ to fit the boundary conditions
Subproblem 2. $X(x) = c_1\cos\sqrt{-\lambda}\,x + c_2\sin\sqrt{-\lambda}\,x$ (for $\lambda<0$)*
$X(0)=c_1=0 \quad X'(\tfrac{\pi}{2}) = \sqrt{-\lambda}\cos\sqrt{-\lambda}\,\tfrac{\pi}{2}=0$
$\Rightarrow \sqrt{-\lambda}=2n+1 \quad X_n(x)=\sin(2n+1)x \quad n=0,1,2,\ldots$
*(note: since $X_0(x)=\sin x$  has no interior zeros, this must be the first eigenfunction & we don't have to check $\lambda\ge 0$.)

$$\psi_2(x,t) = \sum_{n=0}^{\infty} a_n \sin(2n+1)x\; e^{-(2n+1)^2 t}$$

Initial condition:

$$\psi_2(x,0) = \underbrace{\cancel{\cos x}}_{f(x)} - 1 = \sum_{n=0}^{\infty} a_n \sin(2n+1)x$$

$$a_n = \frac{\langle \overset{f(x)}{\cancel{\cos x}}-1,\ \sin(2n+1)x\rangle}{\|\sin(2n+1)x\|^2}$$

§8.2 #5 Let $\Psi = \int_0^\infty e^{-st}\,\psi(x,t)\,dt = \Psi(x,s) = \mathcal{L}(\psi)$

~~Let $F(s) = \int_0^\infty e \quad dx$~~

$$s\,\Psi(x,s) - f(x) = \frac{\partial^2}{\partial x^2}\Psi(x,s) \quad \text{or} \quad \frac{\partial^2}{\partial x^2}\Psi(x,s) - s\,\Psi(x,s) = f(x)$$

This is an ODE with boundary conditions

$$\Psi(0,s) = \mathcal{L}(0) = 0, \quad \Psi(1,s) = \mathcal{L}(5) = \frac{5}{s}$$

We can use the methods from sec. 1.4 or the Green's fcn. approach. We take the latter. First we have to use subproblems because there are 2 nonhomogeneities.

Subproblem 1. $\Psi_1'' - s\Psi_1 = 0, \quad \Psi_1(0) = 0, \quad \Psi_1(1) = 5/s.$

Solution: $\Psi_1(x,s) = \dfrac{5}{s}\dfrac{\sinh\sqrt{s}\,x}{\sinh\sqrt{x}}$

Subproblem 2. $M\Psi_2 \equiv \Psi_2'' - s\Psi_2 = f(x), \quad \Psi_2(0=0), \quad \Psi_2(1)=0$

Similar to sec. 8.1 example 1, the eigenfunctions are $\sin n\pi x$ and the eigenvalues are $-n^2\pi^2 - s$.

$$G(x;x') = \sum_{n=1}^{\infty} \frac{\sin n\pi x \sin n\pi x'}{(-n^2\pi^2 - s)\,\underbrace{\|\sin n\pi x\|^2}_{=1/2}}$$

So $\Psi_2(x,s) = \displaystyle\int_0^1 G(x;x')\,f(x')\,dx$

and $\Psi(x,s) = \Psi_1(x,s) + \Psi_2(x,s).$

Assignments. Don't hand in.

1. One solution of the equation

$$y'' - 4y = \sin x$$

is $-0.2 \sin x$. Use shifted cosh and sinh to find a solution meeting the initial conditions $y(3) = 2$, $y'(3) = 1$.

Hand in your answer on one sheet of paper.

Answer

$y(x) = [2 + .2 \sin 3] \cosh 2(x-3) + 0.5 [1 + .2 \cos 3] \sinh 2(x-3) - .2 \sin x$

or $2.0282 \cosh 2(x-3) + .401 \sinh 2(x-3) - .2 \sin x$

If you got

$2.0105 \cosh 2(x-3) + .599 \sinh 2(x-3) - .2 \sin x$,

you used degrees, not radians.

2. Page 15, # 1[a,b,c,e,f], #2

Page 30 #10[a,b,c]

Page 82 #2

3. Use the software to solve

Example 1, p. 311

Example 1, p. 311, with the Neumann conditions changed to Dirichlet conditions

Laplace's equation in a square, homogen. Dirichlet on the left and bottom, homog. Neumann on the right, nonhomog. Dirichlet on the top.

Example 1, p. 333

4. #1 on page 303. (This equation is not included in USFKAD.)

For those of you who are curious about the nonexistence of a steady state solution to the Neumann problem, it is discussed at length in #10, page 304. It can be really nasty in engineering situations!

5. Page 315: 1 and 3

Page 303: 4.

Readings for EGN 5422, from *PDEs: Sources and Solutions* by A D Snider

Readings to Go with the Lectures

Jan 6 (Week 1A): 1.1

Jan 8 (Week 1B): 1.1

Jan 13 (Week 2A): 1.2

Jan 15 (Week 2B): 2.2, 2.3; also Nagel, Saff, and Snider, *Fundamentals of DEs*, "Qualitative Considerations for Variable-Coefficient and Nonlinear Equations"

Jan 20 (Week 3A): 2.4, 2.5

Jan 22 (Week 3B): 2.5, 3.1-3.4

Jan 27: 3.5, 4.3

Jan 29: 4.3

Feb 3: 4.3-4.5

Feb 5: 4.6

Feb 10: 5.1

Feb 12: 5.2

Feb 24: 5.2, 5.3

Feb 26: 5.3

Mar 2: 6.1

Mar 4: 6.2

Mar 16: 6.3

Mar 18: 6.2, 6.3

Mar 23: 6.3

Mar 25: 6.3, 6.4

Mar 30: 6.4

Apr 1: 7.1

Apr 6: 7.2

Apr 8: 7.3

Apr 13: 1.3, 1.4, 8.1, 8.2

Apr 15: 8.2

Template for solutions to Laplace's equation in a square, in 2 dimensions.

$$[a_1 \cosh kx + a_2 \sinh kx] \times [d_1 \cos ky + d_2 \sin ky]$$

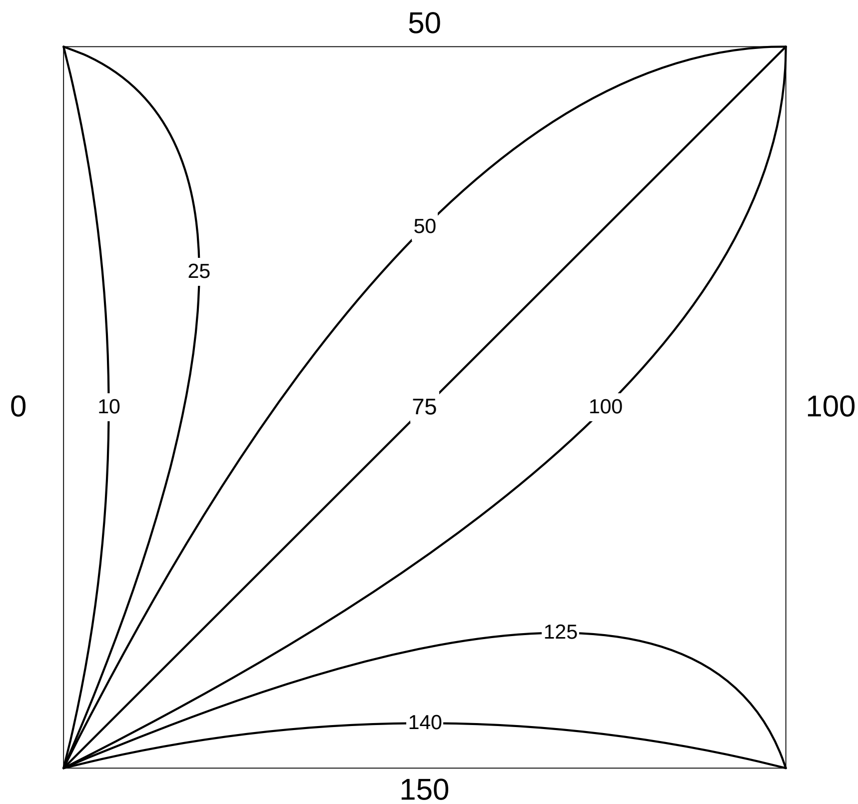
$$+$$

$$[b_1 + b_2 x] \times [e_1 + e_2 y]$$

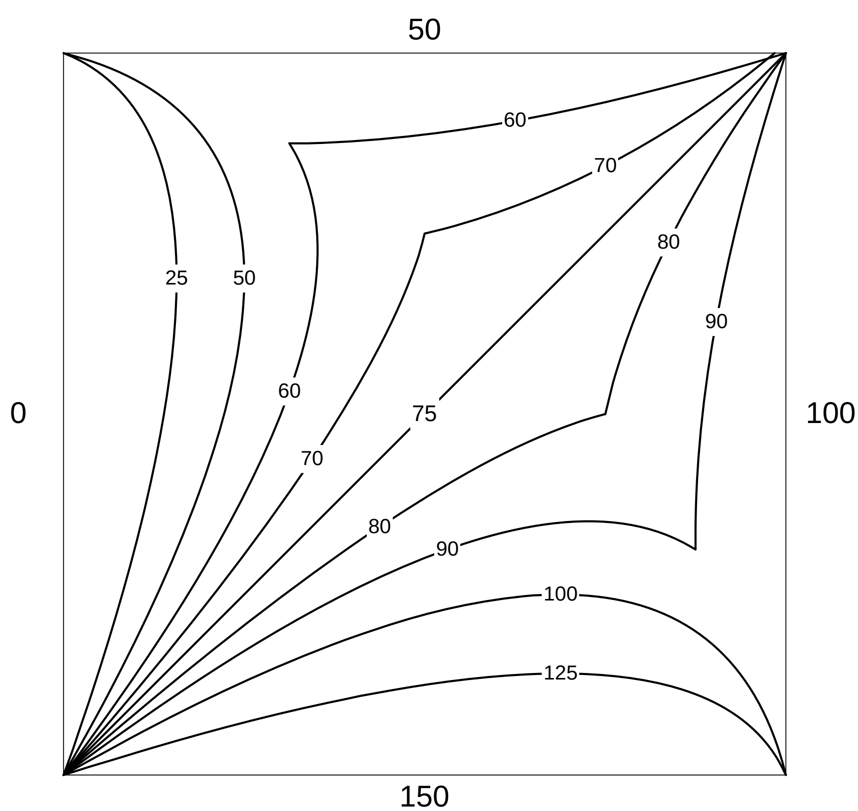
$$+$$

$$[c_1 \cos kx + c_2 \sin kx] \times [f_1 \cosh ky + f_2 \sinh ky]$$

Isotherms



Dave's Solution



Philip's Solution

D'Alembert's Solution

p.237 §4.3 #3

$$\frac{\partial^2 \psi}{\partial t^2} - v^2 \frac{\partial^2 \psi}{\partial x^2} = h(x,t)$$

$$\psi = \int_{\tau=-\infty}^{t} \int_{\xi = x - v(t-\tau)}^{x+v(t-\tau)} \frac{h(\xi,\tau)}{2v}\, d\xi\, d\tau = \int_{\tau=-\infty}^{t} \boxed{}\, d\tau$$

$$\frac{\partial \psi}{\partial t} = \boxed{\Big|_{\tau=t}} + \int_{\tau=-\infty}^{t} \frac{\partial}{\partial t}\boxed{}\, d\tau = \cancel{\int_{\xi=x-v\cdot 0}^{x+v\cdot 0}}$$

$$\int_{\xi=x-v\cdot 0}^{x+v\cdot 0} \frac{h(\xi,t)}{2v}\, d\xi + \int_{\tau=-\infty}^{t} d\tau\, \frac{\partial}{\partial t} \boxed{\int_{\xi=x-v(t-\tau)}^{x+v(t-\tau)} \frac{h(\xi,\tau)}{2v}\, d\xi}$$ (apply Leibnitz again)

$$\int_x^x = 0 + \int_{\tau=-\infty}^{t} d\tau \left[\frac{h(x+v(t-\tau),\tau)}{2v}\cdot v - \frac{h(x-v(t-\tau),\tau)}{2v}\cdot(-v) \right]$$

$$= \frac{1}{2} \int_{\tau=-\infty}^{t} \left[h(x+v(t-\tau),\tau) + h(x-v(t-\tau),\tau) \right] d\tau$$

$$\frac{\partial^2 \psi}{\partial t^2} = \frac{1}{2}\frac{\partial}{\partial t}\int_{\tau=-\infty}^{t} \boxed{}\, d\tau = \frac{1}{2}\boxed{\Big|_{\tau=t}} + \frac{1}{2}\int_{\tau=-\infty} d\tau\, \frac{\partial}{\partial t}\boxed{}$$ again

$$= \frac{1}{2}\left[h(x+v\cdot 0, t) + h(x - v\cdot 0, t) \right]$$

$$+ \frac{1}{2}\int_{\tau=-\infty}^{t} d\tau \left[\frac{\partial h(\xi,\tau)}{\partial \xi}\Big|_{\xi = x+v(t-\tau)} \cdot v + \frac{\partial h(\xi,\tau)}{\partial \xi}\Big|_{\xi = x - v(t-\tau)} \cdot (-v) \right]$$

$$= h(x,t) + \frac{v}{2}\int_{\tau=-\infty}^{t} \left[\frac{\partial h(\xi,\tau)}{\partial \xi}\Big|_{\xi=x+v(t-\tau)} - \frac{\partial h(\xi,\tau)}{\partial \xi}\Big|_{\xi=x-v(t-\tau)} \right] d\tau$$

p.237 §4.3 #3 continued

$$\frac{\partial \psi}{\partial x} = \int_{\tau=-\infty}^{t} d\tau \frac{\partial}{\partial x} \int_{x-v(t-\tau)}^{x+v(t-\tau)} \frac{h(\xi,\tau)}{2v} d\xi = \frac{1}{2v}\int_{\tau=-\infty}^{t} d\tau \left[h(x+v(t-\tau),\tau)\cdot(1) - h(x-v(t-\tau),\tau) \right]$$

$$= \frac{1}{2v}\int_{\tau=-\infty}^{t} \left[h(x+v(t-\tau),\tau) - h(x-v(t-\tau),\tau) \right] d\tau .$$

$$\frac{\partial^2 \psi}{\partial x^2} = \frac{1}{2v}\int_{\tau=-\infty}^{t} \left[\left.\frac{\partial h(\xi,\tau)}{\partial \xi}\right|_{\xi=x-v(t-\tau)} \cdot (1) - \left.\frac{\partial h(\xi,\tau)}{\partial \xi}\right|_{\xi=x-v(t-\tau),\tau} \right] d\tau$$

$$\frac{\partial^2 \psi}{\partial t^2} - v^2 \frac{\partial^2 \psi}{\partial x^2} = h(x,t) + \frac{v}{2}\int_{\tau=-\infty}^{t} \left[\left.\frac{\partial h(\xi,\tau)}{\partial \xi}\right|_{\xi=x+v(t-\tau)} - \left.\frac{\partial h(\xi,\tau)}{\partial \xi}\right|_{\xi=x-v(t-\tau)} \right] d\tau$$

$$- v^2 \cdot \frac{1}{2v} \int_{\tau=-\infty}^{t} \left[\left.\frac{\partial h(\xi,\tau)}{\partial \xi}\right|_{\xi=x+v(t-\tau)} - \left.\frac{\partial h(\xi,\tau)}{\partial \xi}\right|_{\xi=x-v(t-\tau)} \right] d\tau$$

$$= h(x,t) \qquad !!!$$

p. 237 §4.3 #4

According to (10) on p. 234, the solution is

$$\psi(x,t) = \frac{F(x-vt)+F(x+vt)}{2} + \frac{1}{2v}\int_{x-vt}^{x+vt} G(\xi)\,d\xi$$

If we don't want waves traveling to the right, the terms in $(x-vt)$ must cancel; so

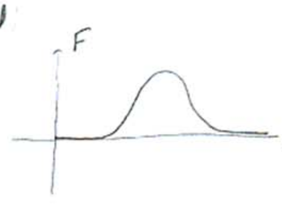
$\frac{F(x-vt)}{2}$ must cancel $\frac{-\mathcal{G}(x-vt)}{2v}$, where $\mathcal{G}'(\xi)=G(\xi)$.

In other words, $F(x-vt) = \frac{\mathcal{G}(x-vt)}{v}$, or $F(\xi) = \frac{\mathcal{G}(\xi)}{v}$, so this implies

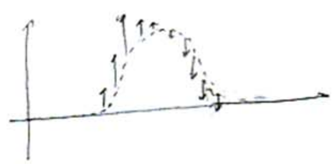
$$\boxed{F'(\xi) = \frac{G(\xi)}{v}}.$$

Interpreting this in terms of (4), the initial velocity at each point must equal v times the initial slope at each point.

If the initial profile F looks like



, then the velocity profile should look like



Think about it; this wave is moving to the left!

Answers for Section 5.2

Answers to Sec. 5.2 (p.1)

(1a) $\dfrac{4\sin 3x \sinh 3y}{\sinh 3\pi}$ (1b) $\dfrac{3\sin 2x \sinh 2y}{\sinh 2\pi} - \dfrac{2\sin 3x \sinh 3y}{\sinh 3\pi}$

(1c) 0 (1d) $\dfrac{5}{8}\dfrac{\sin x \sinh y}{\sinh \pi} - \dfrac{5}{16}\dfrac{\sin 3x \sinh 3y}{\sinh 3\pi} + \dfrac{1}{16}\dfrac{\sin 5x \sinh 5y}{\sinh 5\pi}$

(1e) $\dfrac{\sin x \sinh y}{\cosh \pi}$ (1f) $\dfrac{3\sin 2x \sinh 2y}{2\cosh 2\pi} - \dfrac{2\sin 3x \sinh 3y}{3\cosh 3\pi}$

(2a) $\dfrac{\cos x \cosh y}{\cosh \pi}$ (2b) 1 (2c) 0 (2d) $\dfrac{3\cos 2x \cosh 2y}{\cosh 2\pi} - \dfrac{2\cos 3x \cosh 3y}{\cosh 3\pi}$

(2e) $\dfrac{3\cos 2x \cosh 2y}{2\sinh 2\pi} - \dfrac{2\cos 3x \cosh 3y}{3\sinh 3\pi}$

(5) $\psi = \sum_{n=1}^{\infty} a_n \sin nx \sinh ny$ $\quad \left.\dfrac{\partial\psi}{\partial y} - \psi\right|_{y=\pi} = \sum_{n=1}^{\infty} a_n \sin nx \left[n\cosh n\pi - \sinh n\pi\right] = x(\pi - x)$

so $a_n = \dfrac{\int_0^\pi x(\pi-x)\sin nx\,dx}{\int_0^\pi \sin^2 nx\,dx\,\left[n\cosh n\pi - \sinh n\pi\right]}$ ← $\pi/2$

(6) $\psi = T_0$ (7) no solution

(8) $\psi = \sum_{n=1}^{\infty} a_n \sin nx \cosh ny$ $\quad \psi(x,\pi) = \sum a_n \sin nx \cosh n\pi = 1$

so $a_n = \dfrac{\int_0^\pi 1 \sin nx\,dx}{\int_0^\pi \sin^2 nx\,dx \,\cosh n\pi}$ ← $\pi/2$

(9) $\psi = a_0 x + \sum_{n=1}^{\infty} a_n \sinh nx \cos ny$ $\quad \psi(\pi, y) = a_0\pi + \sum a_n \sinh n\pi \cos ny$

so $a_0 = \dfrac{\int_0^\pi 1\cdot 1\,dy}{\int_0^\pi 1^2\,dy\;\pi} = \dfrac{1}{\pi}$ $\quad a_n = \dfrac{\int_0^\pi 1\cos ny\,dy}{\int_0^\pi \cos^2 ny\,dy\,\sinh n\pi} = 0$

$\boxed{\psi = x/\pi}$

(11) $X = \sin nx \quad Y = c_1 \cosh ny + c_2 \sinh ny$

at $y=0$, $Y' = 2Y$ so $n c_2 = 2c_1$. Choose $Y = n \cosh ny + 2 \sinh ny$.

$$\psi = \sum_{n=1}^{\infty} a_n \sin nx \,[n \cosh ny + 2 \sinh ny]$$

$$\psi(x,\pi) = 1 = \sum a_n \sin nx \,[n \cosh n\pi + 2 \sinh n\pi] \quad \text{so } a_n = \frac{\int_0^\pi 1 \sin nx \, dx}{\underbrace{\int_0^\pi \sin^2 nx \, dx}_{\pi/2} \,[n \cosh n\pi + 2 \sinh n\pi]}$$

(12) $Y = 1, \cos ny \qquad X = a + bx, \quad c \cosh nx + d \sinh nx$

at $x=0$ $\quad X' = X \qquad X' = X$

$b = a \qquad nd = c$

$X = 1 + x \qquad X = n \cosh nx + \sinh nx$

$$\psi = a_0 (1+x) + \sum_{n=1}^{\infty} a_n \cos ny \,[n \cosh nx + \sinh nx]$$

at $x = \pi$, $\frac{\partial \psi}{\partial x} - 3\psi = 1 = a_0[1 - 3(1+\pi)] + \sum_{n=1}^{\infty} a_n \cos ny \,[n^2 \sinh n\pi + n \cosh n\pi - 3n \cosh n\pi - 3 \sinh n\pi]$

$$\text{so } a_0 = \frac{\int_0^\pi 1 \cdot 1 \, dx}{\int_0^\pi 1^2 \, dx \,[-2-3\pi]} \qquad a_n = \frac{\int_0^\pi 1 \cos ny \, dy}{\int_0^\pi \cos^2 ny \, dy \,[\;\;]} = 0$$

$$= \frac{\pi}{\pi\,[-2-3\pi]}$$

$$\boxed{\psi = \frac{-1}{2+3\pi}(1+x)}$$

(13) $$\psi = \sum_{n=1}^{\infty} a_n \sin \frac{n\pi}{b} y \sinh \frac{n\pi}{b}(a-x)$$

$$\psi(0,y) = T_0 = \sum a_n \sin \frac{n\pi}{b} y \sinh \frac{n\pi}{b} a$$

$$\text{So} \quad a_n = \frac{\int_0^b T_0 \sin \frac{n\pi y}{b} \, dy}{\underbrace{\int_0^b \sin^2 \frac{n\pi y}{b} \, dy}_{b/2} \sinh \frac{n\pi a}{b}}$$

Some Fourier Demo Sites

http://www.math.ethz.ch/~lanford/MMP/Fourier/demo3.html

http://www.falstad.com/fourier/

http://www.jhu.edu/~signals/fourier2/

http://www.math.ubc.ca/~feldman/demos/demo3.html

http://homepages.gac.edu/~huber/fourier/

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5 + \Psi_6$$

$\Psi_1 = \sum_{\kappa_y} \sum_{\kappa_z} \cos \kappa_y y \sin \kappa_z z \;\; \eta_x(x; \sqrt{\kappa_y^2 + \kappa_z^2}) \; A(\kappa_y, \, \kappa_z)$
where
$\kappa_y = 0, \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$
$\kappa_z = \frac{\pi}{Z}, \frac{2\pi}{Z}, \frac{3\pi}{Z}, \ldots$

$$\eta_x(x; \sqrt{\kappa_y^2 + \kappa_z^2}) = \begin{cases} X - x & \text{if } \sqrt{\kappa_y^2 + \kappa_z^2} = 0; \\ \sinh \sqrt{\kappa_y^2 + \kappa_z^2}(X - x) & \text{otherwise.} \end{cases}$$

$$A(\kappa_y, \, \kappa_z) = \int_0^Y \, dy \int_0^Z \, dz \;\; \cos \kappa_y y \; N_{\kappa_y} \;\; \sin \kappa_z z \; \tfrac{2}{Z} \; M_{\sqrt{\kappa_y^2 + \kappa_z^2}} \; f_{x=0}(y, z)$$

$$N_{\kappa_y} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_y = 0; \\ \frac{2}{Y} & \text{otherwise} \end{cases}$$

$$M_{\sqrt{\kappa_y^2 + \kappa_z^2}} = \begin{cases} \frac{1}{X} & \text{if } \sqrt{\kappa_y^2 + \kappa_z^2} = 0; \\ \frac{1}{\sinh \sqrt{\kappa_y^2 + \kappa_z^2} X} & \text{otherwise.} \end{cases}$$

$\Psi_2 = \sum_{\kappa_x} \sum_{\kappa_z} \sin \kappa_x x \sin \kappa_z z \;\; \cosh \sqrt{\kappa_x^2 + \kappa_z^2}(Y - y) \; A(\kappa_x, \, \kappa_z)$
where
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\kappa_z = \frac{\pi}{Z}, \frac{2\pi}{Z}, \frac{3\pi}{Z}, \ldots$

$$A(\kappa_x, \, \kappa_z) = \int_0^X \, dx \int_0^Z \, dz \;\; \sin \kappa_x x \; \tfrac{2}{X} \;\; \sin \kappa_z z \; \tfrac{2}{Z} \; M_{\sqrt{\kappa_x^2 + \kappa_z^2}} \; f_{y=0}(x, z)$$

$$M_{\sqrt{\kappa_x^2 + \kappa_z^2}} = \begin{cases} 0 & \text{if } \sqrt{\kappa_x^2 + \kappa_z^2} = 0; \\ \frac{1}{\sqrt{\kappa_x^2 + \kappa_z^2} \sinh \sqrt{\kappa_x^2 + \kappa_z^2} Y} & \text{otherwise.} \end{cases}$$

$$\Psi_3 = \sum_{\kappa_x} \sum_{\kappa_y} \sin \kappa_x x \cos \kappa_y y \;\; \eta_z(z; \sqrt{\kappa_x^2 + \kappa_y^2}) \; A(\kappa_x, \, \kappa_y)$$

where
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\kappa_y = 0, \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$

$$\eta_z(z; \sqrt{\kappa_x^2+\kappa_y^2}) = \begin{cases} Z - z & \text{if } \sqrt{\kappa_x^2+\kappa_y^2} = 0; \\ \sinh \sqrt{\kappa_x^2+\kappa_y^2}(Z-z) & \text{otherwise.} \end{cases}$$

$$A(\kappa_x,\, \kappa_y) = \int_0^X \, dx \int_0^Y \, dy \, \sin \kappa_x x \, \tfrac{2}{X} \, \cos \kappa_y y \, N_{\kappa_y} \, M_{\sqrt{\kappa_x^2+\kappa_y^2}} \, f_{z=0}(x,y)$$

$$N_{\kappa_y} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_y = 0; \\ \frac{2}{Y} & \text{otherwise} \end{cases}$$

$$M_{\sqrt{\kappa_x^2+\kappa_y^2}} = \begin{cases} \frac{1}{Z} & \text{if } \sqrt{\kappa_x^2+\kappa_y^2} = 0; \\ \frac{1}{\sinh \sqrt{\kappa_x^2+\kappa_y^2} Z} & \text{otherwise.} \end{cases}$$

$$\Psi_4 = \sum_{\kappa_y} \sum_{\kappa_z} \cos \kappa_y y \sin \kappa_z z \; \eta_x(x; \sqrt{\kappa_y^2+\kappa_z^2}) \; A(\kappa_y,\, \kappa_z)$$

where
$\kappa_y = 0, \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$
$\kappa_z = \frac{\pi}{Z}, \frac{2\pi}{Z}, \frac{3\pi}{Z}, \ldots$

$$\eta_x(x; \sqrt{\kappa_y^2+\kappa_z^2}) = \begin{cases} x & \text{if } \sqrt{\kappa_y^2+\kappa_z^2} = 0; \\ \sinh \sqrt{\kappa_y^2+\kappa_z^2} x & \text{otherwise.} \end{cases}$$

$$A(\kappa_y,\, \kappa_z) = \int_0^Y \, dy \int_0^Z \, dz \, \cos \kappa_y y \, N_{\kappa_y} \, \sin \kappa_z z \, \tfrac{2}{Z} \, M_{\sqrt{\kappa_y^2+\kappa_z^2}} \, f_{x=X}(y,z)$$

$$N_{\kappa_y} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_y = 0; \\ \frac{2}{Y} & \text{otherwise} \end{cases}$$

$$M_{\sqrt{\kappa_y^2+\kappa_z^2}} = \begin{cases} \frac{1}{X} & \text{if } \sqrt{\kappa_y^2+\kappa_z^2} = 0, \\ \frac{1}{\sinh \sqrt{\kappa_y^2+\kappa_z^2} X} & \text{otherwise.} \end{cases}$$

$$\Psi_5 = \sum_{\kappa_x} \sum_{\kappa_z} \sin \kappa_x x \sin \kappa_z z \; \cosh \sqrt{\kappa_x^2+\kappa_z^2} y \; A(\kappa_x,\, \kappa_z)$$

where
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\kappa_z = \frac{\pi}{Z}, \frac{2\pi}{Z}, \frac{3\pi}{Z}, \ldots$

$$A(\kappa_x,\, \kappa_z) = \int_0^X \, dx \int_0^Z \, dz \, \sin \kappa_x x \, \tfrac{2}{X} \, \sin \kappa_z z \, \tfrac{2}{Z} \, M_{\sqrt{\kappa_x^2+\kappa_z^2}} \, f_{y=Y}(x,z)$$

$$M_{\sqrt{\kappa_x^2+\kappa_z^2}} = \begin{cases} 0 & \text{if } \sqrt{\kappa_x^2+\kappa_z^2}=0; \\ \frac{1}{\sqrt{\kappa_x^2+\kappa_z^2}\sinh\sqrt{\kappa_x^2+\kappa_z^2}Y} & \text{otherwise.} \end{cases}$$

$\Psi_6 = \sum_{\kappa_x}\sum_{\kappa_y} \sin\kappa_x x \cos\kappa_y y \; \eta_z(z; \sqrt{\kappa_x^2+\kappa_y^2}) \; A(\kappa_x,\, \kappa_y)$
$where$
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\kappa_y = 0, \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$
$\eta_z(z; \sqrt{\kappa_x^2+\kappa_y^2}) = \begin{cases} z & \text{if } \sqrt{\kappa_x^2+\kappa_y^2}=0; \\ \sinh\sqrt{\kappa_x^2+\kappa_y^2}z & \text{otherwise.} \end{cases}$
$A(\kappa_x,\, \kappa_y) = \int_0^X \; dx \int_0^Y \; dy \;\; \sin\kappa_x x \; \frac{2}{X} \;\; \cos\kappa_y y \; N_{\kappa_y} \; M_{\sqrt{\kappa_x^2+\kappa_y^2}} \; f_{z=Z}(x,y)$

$N_{\kappa_y} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_y = 0; \\ \frac{2}{Y} & \text{otherwise} \end{cases}$

$M_{\sqrt{\kappa_x^2+\kappa_y^2}} = \begin{cases} \frac{1}{Z} & \text{if } \sqrt{\kappa_x^2+\kappa_y^2}=0, \\ \frac{1}{\sinh\sqrt{\kappa_x^2+\kappa_y^2}Z} & \text{otherwise.} \end{cases}$

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3$$

$$\Psi_1 = \sum_{\kappa_\theta=-\infty}^{\infty} \sum_{\kappa_{r:\theta}} \; e^{i\kappa_\theta \theta} \; J_{\kappa_\theta}(\kappa_{r:\theta} r) \;\; \eta_z(z; \kappa_{r:\theta}) \; A(\kappa_\theta, \kappa_{r:\theta})$$

where

for each value of $\kappa_\theta, \kappa_{r:\theta} = \frac{j_{\kappa_\theta, \kappa_{r:\theta}}}{b}$ where
$\{j_{\kappa_\theta, \kappa_{r:\theta}}\}$ are the positive roots of $J_{\kappa_\theta}(j_{\kappa_\theta, \kappa_{r:\theta}}) = 0$

$$\eta_z(z; \kappa_{r:\theta}) = \begin{cases} Z - z & \text{if } \kappa_{r:\theta} = 0; \\ \sinh \kappa_{r:\theta}(Z - z) & \text{otherwise.} \end{cases}$$

$$A(\kappa_\theta, \kappa_{r:\theta}) = \int_0^{2\pi} d\theta \int_a^b dr \; e^{-i\kappa_\theta \theta} \; \frac{1}{2\pi} \; J_{\kappa_\theta}(\kappa_{r:\theta} r) r \\ \times \; \frac{2}{b^2 {J_{\kappa_\theta+1}}^2(j_{\kappa_\theta, \kappa_{r:\theta}})} \; M_{\kappa_{r:\theta}} \; f_{z=0}(\theta, r)$$

$$M_{\kappa_{r:\theta}} = \begin{cases} \frac{1}{Z} & \text{if } \kappa_{r:\theta} = 0; \\ \frac{1}{\sinh \kappa_{r:\theta} Z} & \text{otherwise.} \end{cases}$$

$$\Psi_2 = \sum_{\kappa_\theta=-\infty}^{\infty} \sum_{\kappa_{r:\theta}} \; e^{i\kappa_\theta \theta} \; J_{\kappa_\theta}(\kappa_{r:\theta} r) \; \eta_z(z; \kappa_{r:\theta}) \; A(\kappa_\theta, \kappa_{r:\theta})$$

where

for each value of $\kappa_\theta, \kappa_{r:\theta} = \frac{j_{\kappa_\theta, \kappa_{r:\theta}}}{b}$ where
$\{j_{\kappa_\theta, \kappa_{r:\theta}}\}$ are the positive roots of $J_{\kappa_\theta}(j_{\kappa_\theta, \kappa_{r:\theta}}) = 0$

$$\eta_z(z; \kappa_{r:\theta}) = \begin{cases} z & \text{if } \kappa_{r:\theta} = 0; \\ \sinh \kappa_{r:\theta} z & \text{otherwise.} \end{cases}$$

$$A(\kappa_\theta, \kappa_{r:\theta}) = \int_0^{2\pi} d\theta \int_a^b dr \; e^{-i\kappa_\theta \theta} \; \frac{1}{2\pi} \; J_{\kappa_\theta}(\kappa_{r:\theta} r) r \\ \times \; \frac{2}{b^2 {J_{\kappa_\theta+1}}^2(j_{\kappa_\theta, \kappa_{r:\theta}})} \; M_{\kappa_{r:\theta}} \; f_{z=Z}(\theta, r)$$

$$M_{\kappa_{r:\theta}} = \begin{cases} \frac{1}{Z} & \text{if } \kappa_{r:\theta} = 0, \\ \frac{1}{\sinh \kappa_{r:\theta} Z} & \text{otherwise.} \end{cases}$$

$\Psi_3 = \sum_{\kappa_\theta=-\infty}^{\infty} \sum_{\kappa_z} \ e^{i\kappa_\theta\theta} \sin\kappa_z z \ \eta_r(r,\kappa_\theta,\kappa_\theta,\kappa_z) \ A(\kappa_\theta,\kappa_z)$
$where$

$\kappa_z = \frac{\pi}{Z}, \frac{2\pi}{Z}, \frac{3\pi}{Z}, \ldots$

$$\eta_r(r,\kappa_\theta,\kappa_\theta,\kappa_z) = \begin{cases} 1 \text{ if } \kappa_\theta = \kappa_\theta, \kappa_z = 0; \\ r^{\kappa_\theta} \text{ if } \kappa_\theta \neq 0 \text{ and } \kappa_\theta, \kappa_z = 0; \text{ otherwise} \\ I_{\kappa_\theta}(\kappa_\theta, \kappa_z r) \end{cases}$$

$A(\kappa_\theta,\kappa_z) = \int_0^{2\pi} \ d\theta \int_0^Z \ dz \ e^{-i\kappa_\theta\theta} \ \frac{1}{2\pi} \ \sin\kappa_z z \ \frac{2}{Z} \ N_r \ f_{r=b}(\theta,z)$

$$N_r = \begin{cases} 1 & \text{if } \kappa_\theta = \kappa_\theta, \kappa_z = 0; \\ b^{-\kappa_\theta} & \text{if } \kappa_\theta \neq 0 \text{ and } \kappa_\theta, \kappa_z = 0; \\ \frac{1}{I_{\kappa_\theta}(\kappa_\theta,\kappa_z b)} & \text{if } \kappa_\theta \neq 0 \text{ and } \kappa_\theta, \kappa_z \neq 0. \end{cases}$$

$$\Psi = \Psi_1$$

$$\Psi_1 = \sum_{\kappa_z} \int_0^\infty \, d\kappa_{r:z} \sin\kappa_z z \left[K_{i\kappa_{r:z}}(\kappa_z b) I_{i\kappa_{r:z}}(\kappa_z r) - I_{i\kappa_{r:z}}(\kappa_z b) K_{i\kappa_{r:z}}(\kappa_z r)\right]$$
$$\times \quad \eta_\theta(\theta;\kappa_{r:z}) \; A(\kappa_z,\kappa_{r:z})$$

$where$

$$\kappa_z = \tfrac{\pi}{Z}, \tfrac{2\pi}{Z}, \tfrac{3\pi}{Z}, \ldots$$

$$\eta_\theta(\theta;\kappa_{r:z}) = \begin{cases} \theta & \text{if } \kappa_{r:z} = 0; \\ \sinh\kappa_{r:z}\theta & \text{otherwise.} \end{cases}$$

$$A(\kappa_z,\kappa_{r:z}) = \int_0^Z \, dz \int_0^b \, dr \; \sin\kappa_z z \; \tfrac{2}{Z}$$
$$\times \; \left[K_{i\kappa_{r:z}}(\kappa_z b) I_{i\kappa_{r:z}}(\kappa_z r) - I_{i\kappa_{r:z}}(\kappa_z b) K_{i\kappa_{r:z}}(\kappa_z r)\right] \tfrac{1}{r}$$
$$\times \quad \tfrac{2\kappa_{r:z}\sinh\kappa_{r:z}\pi}{\pi^2 |I_{i\kappa_{r:z}}(\kappa_z b)|^2} \; M_{\kappa_{r:z}} \; f_{\theta=\Theta}(z,r)$$

$$M_{\kappa_{r:z}} = \begin{cases} \frac{1}{\Theta} & \text{if } \kappa_{r:z} = 0, \\ \frac{1}{\sinh\kappa_{r:z}\Theta} & \text{otherwise.} \end{cases}$$

$$\Psi = \Psi_1$$

$\Psi_1 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\phi, \theta)\ r^{\ell}\ A_{\ell m}$
$where$

$$A_{\ell m} = \int_0^{\pi}\ \sin\phi\, d\phi \int_0^{2\pi}\ d\theta\ Y^*_{\ell m}(\phi, \theta) \quad M_r\ f_{r=b}(\theta, \phi)$$

$$M_r = \frac{1}{b^{\ell}}$$

$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4$$

$\Psi_1 = \sum_{\kappa_y} \sin \kappa_y y \;\; \eta_x(x;\kappa_y)\; A(\kappa_y)$
where
$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$
$\eta_x(x;\kappa_y) = \begin{cases} X - x & \text{if } \kappa_y = 0; \\ \sinh \kappa_y (X - x) & \text{otherwise.} \end{cases}$
$A(\kappa_y) = \int_0^Y \, dy \;\, \sin \kappa_y y \, \frac{2}{Y} \, M_{\kappa_y} \; f_{x=0}(y)$

$$M_{\kappa_y} = \begin{cases} \frac{1}{X} & \text{if } \kappa_y = 0; \\ \frac{1}{\sinh \kappa_y X} & \text{otherwise.} \end{cases}$$

$\Psi_2 = \sum_{\kappa_x} \sin \kappa_x x \;\; \eta_y(y;\kappa_x)\; A(\kappa_x)$
where
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\eta_y(y;\kappa_x) = \begin{cases} Y - y & \text{if } \kappa_x = 0; \\ \sinh \kappa_x (Y - y) & \text{otherwise.} \end{cases}$
$A(\kappa_x) = \int_0^X \, dx \;\, \sin \kappa_x x \, \frac{2}{X} \, M_{\kappa_x} \; f_{y=0}(x)$

$$M_{\kappa_x} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_x = 0; \\ \frac{1}{\sinh \kappa_x Y} & \text{otherwise.} \end{cases}$$

$\Psi_3 = \sum_{\kappa_y} \sin \kappa_y y \; \eta_x(x;\kappa_y)\; A(\kappa_y)$
where
$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$

$$\eta_x(x;\kappa_y) = \begin{cases} x & \text{if } \kappa_y = 0; \\ \sinh \kappa_y x & \text{otherwise.} \end{cases}$$
$$A(\kappa_y) = \int_0^Y dy\ \sin \kappa_y y\ \tfrac{2}{Y}\ M_{\kappa_y}\ f_{x=X}(y)$$

$$M_{\kappa_y} = \begin{cases} \frac{1}{X} & \text{if } \kappa_y = 0, \\ \frac{1}{\sinh \kappa_y X} & \text{otherwise.} \end{cases}$$

$$\Psi_4 = \textstyle\sum_{\kappa_x} \sin \kappa_x x\ \eta_y(y;\kappa_x)\ A(\kappa_x)$$
$$where$$
$$\kappa_x = \tfrac{\pi}{X}, \tfrac{2\pi}{X}, \tfrac{3\pi}{X}, \ldots$$
$$\eta_y(y;\kappa_x) = \begin{cases} y & \text{if } \kappa_x = 0; \\ \sinh \kappa_x y & \text{otherwise.} \end{cases}$$
$$A(\kappa_x) = \int_0^X dx\ \sin \kappa_x x\ \tfrac{2}{X}\ M_{\kappa_x}\ f_{y=Y}(x)$$

$$M_{\kappa_x} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_x = 0, \\ \frac{1}{\sinh \kappa_x Y} & \text{otherwise.} \end{cases}$$

$$\Psi_{transient} = \textstyle\sum_{\kappa_x} \sum_{\kappa_y} \sin \kappa_x x \sin \kappa_y y\ e^{-(\kappa_x^2+\kappa_y^2)t}\ A(\kappa_x,\kappa_y)$$
$$where$$
$$\kappa_x = \tfrac{\pi}{X}, \tfrac{2\pi}{X}, \tfrac{3\pi}{X}, \ldots$$
$$\kappa_y = \tfrac{\pi}{Y}, \tfrac{2\pi}{Y}, \tfrac{3\pi}{Y}, \ldots$$
$$A(\kappa_x,\kappa_y) = \int_0^X dx \int_0^Y dy\ \sin \kappa_x x\ \tfrac{2}{X}\ \sin \kappa_y y\ \tfrac{2}{Y} \\ \times\ \left[\Psi(x,y;0) - \Psi_{steady\ state}\right]$$

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$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = \Psi_1$$

$$\Psi_1 = \tfrac{x}{X} f_{x=X}$$

$where$

$=$

$$\Psi_{transient} = \textstyle\sum_{\kappa_x} \sin \kappa_x x \; e^{-\kappa_x^2 t} \; A(\kappa_x)$$

$where$

$$\kappa_x = \tfrac{\pi}{X}, \tfrac{2\pi}{X}, \tfrac{3\pi}{X}, \ldots$$

$$A(\kappa_x) = \textstyle\int_0^X \; dx \;\; \sin \kappa_x x \; \tfrac{2}{X}$$

$$\times \;\; \left[\Psi(x;0) - \Psi_{steady\ state}\right]$$

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$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = \Psi_1$$

$\Psi_1 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\phi, \theta)\ r^{\ell}\ A_{\ell m}$
$where$

$$A_{\ell m} = \int_0^{\pi}\ \sin\phi\, d\phi \int_0^{2\pi}\ d\theta\ Y^*_{\ell m}(\phi, \theta) \quad M_r\ f_{r=b}(\theta, \phi)$$

$$M_r = \frac{1}{b^{\ell}}$$

$\Psi_{transient} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{p=1}^{\infty} Y_{\ell m}(\phi, \theta)\ j_{\ell}(\kappa_{\ell,\,p} r)\ e^{-\kappa^2_{\ell:r} t}\ A_{\ell m}(\kappa_{\ell:r})$
$where$

$\kappa_{\ell,\,p} = s_{\ell,\,p}/b$ and$s_{\ell,\,p}$ is the pth positive zero of j_{ℓ}.
$A_{\ell m}(\kappa_{\ell:r}) = \int_0^{\pi}\ \sin\phi\, d\phi \int_0^{2\pi}\ d\theta \int_0^{b}\ r^2\, dr\ Y^*_{\ell m}(\phi, \theta) \quad j_{\ell}(\kappa_{\ell,\,p} r)\ \frac{2}{b^3 j^2_{\ell+1}(s_{\ell,p})}$
$\times\ \left[\Psi(\theta, \phi, r; 0) - \Psi_{steady\ state}\right]$

page 459 Fig 6

$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = 0$$

$$\Psi_{transient} = \int_0^\infty d\kappa_x \cos\kappa_x x \ e^{-\kappa_x^2 t} \ A(\kappa_x)$$
where

$$A(\kappa_x) = \int_0^\infty dx \ \cos\kappa_x x \ \frac{2}{\pi}$$

$$\times \ [\Psi(x;0) - \Psi_{steady\ state}]$$

page 459 Fig 7

$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = \Psi_1$$

$$\Psi_1 = \ f_{x=0}$$

$where$

$$=$$

$$\Psi_{transient} = \int_0^\infty \ d\kappa_x \sin \kappa_x x \ e^{-\kappa_x^2 t} \ A(\kappa_x)$$

$where$

$$A(\kappa_x) = \int_0^\infty \ dx \ \ \sin \kappa_x x \ \frac{2}{\pi}$$

$$\times \ \left[\Psi(x;0) - \Psi_{steady\ state}\right]$$

$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = \Psi_1$$
$$\Psi_{transient} = \Psi_{transient\ \#1} + \Psi_{transient\ \#2}$$

$$\Psi_1 = \ \frac{x}{X} f_{x=X}$$
$where$

$=$

$$\Psi_{transient\ \#1} = \sum_{\kappa_x} \sin \kappa_x x\ \ \cos \kappa_x t\ A(\kappa_x)$$
$where$
$$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$$
$$A(\kappa_x) = \int_0^X\ dx\ \ \sin \kappa_x x\ \frac{2}{X} \qquad \times\ \left[\Psi(x;0) - \Psi_{steady\ state}\right]$$

$$\Psi_{transient\ \#2} = \sum_{\kappa_x} \sin \kappa_x x\ \frac{\sin \kappa_x t}{\kappa_x}\ A(\kappa_x)$$
$where$
$$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$$
$$A(\kappa_x) = \int_0^X\ dx\ \ \sin \kappa_x x\ \frac{2}{X} \qquad \times\ \frac{\partial \Psi(x;0)}{\partial t}$$

$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = 0$$
$$\Psi_{transient} = \Psi_{transient\,\#1} + \Psi_{transient\,\#2}$$

$$\Psi_{transient\,\#1} = \sum_{\kappa_x} \sum_{\kappa_y} \sum_{\kappa_z} \cos\kappa_x x \cos\kappa_y y \cos\kappa_z z \;\; \cos\sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}\,t \;\, A(\kappa_x, \kappa_y, \kappa_z)$$

$where$

$$\kappa_x = 0, \tfrac{\pi}{X}, \tfrac{2\pi}{X}, \tfrac{3\pi}{X}, \ldots$$
$$\kappa_y = 0, \tfrac{\pi}{Y}, \tfrac{2\pi}{Y}, \tfrac{3\pi}{Y}, \ldots$$
$$\kappa_z = 0, \tfrac{\pi}{Z}, \tfrac{2\pi}{Z}, \tfrac{3\pi}{Z}, \ldots$$

$$A(\kappa_x, \kappa_y, \kappa_z) = \int_0^X dx \int_0^Y dy \int_0^Z dz \;\; \cos\kappa_x x \; N_{\kappa_x} \;\; \cos\kappa_y y \; N_{\kappa_y} \;\; \cos\kappa_z z \; N_{\kappa_z}$$
$$\times \;\; [\Psi(x,y,z;0) - \Psi_{steady\ state}]$$

$$N_{\kappa_x} = \begin{cases} \frac{1}{X} & \text{if } \kappa_x = 0; \\ \frac{2}{X} & \text{otherwise} \end{cases}$$
$$N_{\kappa_y} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_y = 0; \\ \frac{2}{Y} & \text{otherwise} \end{cases}$$
$$N_{\kappa_z} = \begin{cases} \frac{1}{Z} & \text{if } \kappa_z = 0; \\ \frac{2}{Z} & \text{otherwise} \end{cases}$$

$$\Psi_{transient\,\#2} = \sum_{\kappa_x} \sum_{\kappa_y} \sum_{\kappa_z} \cos\kappa_x x \cos\kappa_y y \cos\kappa_z z \;\; \frac{\sin\sqrt{\kappa_x^2+\kappa_y^2+\kappa_z^2}\,t}{\sqrt{\kappa_x^2+\kappa_y^2+\kappa_z^2}} \;\, A(\kappa_x, \kappa_y, \kappa_z)$$

$where$

$$\kappa_x = 0, \tfrac{\pi}{X}, \tfrac{2\pi}{X}, \tfrac{3\pi}{X}, \ldots$$
$$\kappa_y = 0, \tfrac{\pi}{Y}, \tfrac{2\pi}{Y}, \tfrac{3\pi}{Y}, \ldots$$
$$\kappa_z = 0, \tfrac{\pi}{Z}, \tfrac{2\pi}{Z}, \tfrac{3\pi}{Z}, \ldots$$

$$A(\kappa_x, \kappa_y, \kappa_z) = \int_0^X dx \int_0^Y dy \int_0^Z dz \;\; \cos\kappa_x x \; N_{\kappa_x} \;\; \cos\kappa_y y \; N_{\kappa_y} \;\; \cos\kappa_z z \; N_{\kappa_z}$$
$$\times \;\; \frac{\partial\Psi(x,y,z;0)}{\partial t}$$

$$N_{\kappa_x} = \begin{cases} \frac{1}{X} & \text{if } \kappa_x = 0; \\ \frac{2}{X} & \text{otherwise} \end{cases}$$
$$N_{\kappa_y} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_y = 0; \\ \frac{2}{Y} & \text{otherwise} \end{cases}$$
$$N_{\kappa_z} = \begin{cases} \frac{1}{Z} & \text{if } \kappa_z = 0; \\ \frac{2}{Z} & \text{otherwise} \end{cases}$$

$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = 0$$
$$\Psi_{transient} = \Psi_{transient\,\#1} + \Psi_{transient\,\#2}$$

$$\Psi_{transient\,\#1} = \textstyle\sum_{\kappa_x} \sum_{\kappa_y} \sum_{\kappa_z} \sin \kappa_x x \sin \kappa_y y \cos \kappa_z z \;\; \cos \sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}\, t \; A(\kappa_x, \kappa_y, \kappa_z)$$
$where$
$$\kappa_x = \tfrac{\pi}{X}, \tfrac{2\pi}{X}, \tfrac{3\pi}{X}, \ldots$$
$$\kappa_y = \tfrac{\pi}{Y}, \tfrac{2\pi}{Y}, \tfrac{3\pi}{Y}, \ldots$$
$$\kappa_z = 0, \tfrac{\pi}{Z}, \tfrac{2\pi}{Z}, \tfrac{3\pi}{Z}, \ldots$$
$$A(\kappa_x, \kappa_y, \kappa_z) = \textstyle\int_0^X dx \int_0^Y dy \int_0^Z dz \;\sin \kappa_x x \;\tfrac{2}{X}\; \sin \kappa_y y \;\tfrac{2}{Y}\; \cos \kappa_z z \; N_{\kappa_z} \times \left[\Psi(x, y, z; 0) - \Psi_{steady\ state}\right]$$

$$N_{\kappa_z} = \begin{cases} \frac{1}{Z} & \text{if } \kappa_z = 0; \\ \frac{2}{Z} & \text{otherwise} \end{cases}$$

$$\Psi_{transient\,\#2} = \textstyle\sum_{\kappa_x} \sum_{\kappa_y} \sum_{\kappa_z} \sin \kappa_x x \sin \kappa_y y \cos \kappa_z z \; \frac{\sin \sqrt{\kappa_x^2+\kappa_y^2+\kappa_z^2}\,t}{\sqrt{\kappa_x^2+\kappa_y^2+\kappa_z^2}} \; A(\kappa_x, \kappa_y, \kappa_z)$$
$where$
$$\kappa_x = \tfrac{\pi}{X}, \tfrac{2\pi}{X}, \tfrac{3\pi}{X}, \ldots$$
$$\kappa_y = \tfrac{\pi}{Y}, \tfrac{2\pi}{Y}, \tfrac{3\pi}{Y}, \ldots$$
$$\kappa_z = 0, \tfrac{\pi}{Z}, \tfrac{2\pi}{Z}, \tfrac{3\pi}{Z}, \ldots$$
$$A(\kappa_x, \kappa_y, \kappa_z) = \textstyle\int_0^X dx \int_0^Y dy \int_0^Z dz \;\sin \kappa_x x \;\tfrac{2}{X}\; \sin \kappa_y y \;\tfrac{2}{Y}\; \cos \kappa_z z \; N_{\kappa_z} \times \frac{\partial \Psi(x,y,z;0)}{\partial t}$$

$$N_{\kappa_z} = \begin{cases} \frac{1}{Z} & \text{if } \kappa_z = 0; \\ \frac{2}{Z} & \text{otherwise} \end{cases}$$

$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = 0$$
$$\Psi_{transient} = \Psi_{transient\,\#1} + \Psi_{transient\,\#2}$$

$$\Psi_{transient\,\#1} = \sum_{\kappa_x} \sum_{\kappa_y} \int_{-\infty}^{\infty} d\kappa_z \sin\kappa_x x \sin\kappa_y y\ e^{i\kappa_z z}\ \cos\sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}\,t\ A(\kappa_x, \kappa_y, \kappa_z)$$

$where$

$$\kappa_x = \tfrac{\pi}{X}, \tfrac{2\pi}{X}, \tfrac{3\pi}{X}, \ldots$$
$$\kappa_y = \tfrac{\pi}{Y}, \tfrac{2\pi}{Y}, \tfrac{3\pi}{Y}, \ldots$$

$$A(\kappa_x, \kappa_y, \kappa_z) = \int_0^X\ dx \int_0^Y\ dy \int_{-\infty}^{\infty} dz\ \sin\kappa_x x\ \tfrac{2}{X}\ \sin\kappa_y y\ \tfrac{2}{Y}\ e^{-i\kappa_z z}\ \tfrac{1}{2\pi} \times \left[\Psi(x, y, z; 0) - \Psi_{steady\ state}\right]$$

$$\Psi_{transient\,\#2} = \sum_{\kappa_x} \sum_{\kappa_y} \int_{-\infty}^{\infty} d\kappa_z \sin\kappa_x x \sin\kappa_y y\ e^{i\kappa_z z}\ \frac{\sin\sqrt{\kappa_x^2+\kappa_y^2+\kappa_z^2}\,t}{\sqrt{\kappa_x^2+\kappa_y^2+\kappa_z^2}}\ A(\kappa_x, \kappa_y, \kappa_z)$$

$where$

$$\kappa_x = \tfrac{\pi}{X}, \tfrac{2\pi}{X}, \tfrac{3\pi}{X}, \ldots$$
$$\kappa_y = \tfrac{\pi}{Y}, \tfrac{2\pi}{Y}, \tfrac{3\pi}{Y}, \ldots$$

$$A(\kappa_x, \kappa_y, \kappa_z) = \int_0^X\ dx \int_0^Y\ dy \int_{-\infty}^{\infty} dz\ \sin\kappa_x x\ \tfrac{2}{X}\ \sin\kappa_y y\ \tfrac{2}{Y}\ e^{-i\kappa_z z}\ \tfrac{1}{2\pi} \times \frac{\partial\Psi(x,y,z;0)}{\partial t}$$

$$\Psi = \Psi_{steady\;state} + \Psi_{transient}$$

$$\Psi_{steady\;state} = 0$$
$$\Psi_{transient} = \Psi_{transient\,\#1} + \Psi_{transient\,\#2}$$

$$\Psi_{transient\,\#1} = \int_{-\infty}^{\infty} d\kappa_x \int_{-\infty}^{\infty} d\kappa_y \int_{-\infty}^{\infty} d\kappa_z\; e^{i\kappa_x x}\; e^{i\kappa_y y}\; e^{i\kappa_z z}\; \cos\sqrt{\kappa_x^2 + \kappa_y^2 + \kappa_z^2}t\; A(\kappa_x, \kappa_y, \kappa_z)$$
$where$

$$A(\kappa_x, \kappa_y, \kappa_z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\; e^{-i\kappa_x x}\, \tfrac{1}{2\pi}\; e^{-i\kappa_y y}\, \tfrac{1}{2\pi}\; e^{-i\kappa_z z}\, \tfrac{1}{2\pi} \times \left[\Psi(x, y, z; 0) - \Psi_{steady\;state}\right]$$

$$\Psi_{transient\,\#2} = \int_{-\infty}^{\infty} d\kappa_x \int_{-\infty}^{\infty} d\kappa_y \int_{-\infty}^{\infty} d\kappa_z\; e^{i\kappa_x x}\; e^{i\kappa_y y}\; e^{i\kappa_z z}\; \frac{\sin\sqrt{\kappa_x^2+\kappa_y^2+\kappa_z^2}t}{\sqrt{\kappa_x^2+\kappa_y^2+\kappa_z^2}}\; A(\kappa_x, \kappa_y, \kappa_z)$$
$where$

$$A(\kappa_x, \kappa_y, \kappa_z) = \int_{-\infty}^{\infty} dx \int_{-\infty}^{\infty} dy \int_{-\infty}^{\infty} dz\; e^{-i\kappa_x x}\, \tfrac{1}{2\pi}\; e^{-i\kappa_y y}\, \tfrac{1}{2\pi}\; e^{-i\kappa_z z}\, \tfrac{1}{2\pi} \times \frac{\partial\Psi(x,y,z;0)}{\partial t}$$

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$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = 0$$
$$\Psi_{transient} = \Psi_{transient\ \#1} + \Psi_{transient\ \#2}$$

$$\Psi_{transient\ \#1} = \sum_{\kappa_\theta=-\infty}^{\infty} \int_{-\infty}^{\infty} d\kappa_z \sum_{\kappa_{r:\theta}} e^{i\kappa_\theta\theta}\ e^{i\kappa_z z}\ J_{\kappa_\theta}(\kappa_{r:\theta} r) \cos\sqrt{\kappa_{r:\theta}^2 + \kappa_z^2}t\ A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z)$$

where

for each value of κ_θ, $\kappa_{r:\theta} = \frac{j_{\kappa_\theta, \kappa_{r:\theta}}}{b}$ where
$\{j_{\kappa_\theta, \kappa_{r:\theta}}\}$ are the positive roots of $J_{\kappa_\theta}(j_{\kappa_\theta, \kappa_{r:\theta}}) = 0$

$$A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z) = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \int_a^b dr\ e^{-i\kappa_\theta\theta}\ \frac{1}{2\pi}\ e^{-i\kappa_z z}\ \frac{1}{2\pi}\ J_{\kappa_\theta}(\kappa_{r:\theta} r) r \times \frac{2}{b^2 {J_{\kappa_\theta+1}}^2(j_{\kappa_\theta, \kappa_{r:\theta}})} \times [\Psi(r, \theta, z; 0) - \Psi_{steady\ state}]$$

$$\Psi_{transient\ \#2} = \sum_{\kappa_\theta=-\infty}^{\infty} \int_{-\infty}^{\infty} d\kappa_z \sum_{\kappa_{r:\theta}} e^{i\kappa_\theta\theta}\ e^{i\kappa_z z}\ J_{\kappa_\theta}(\kappa_{r:\theta} r) \frac{\sin\sqrt{\kappa_{r:\theta}^2 + \kappa_z^2}t}{\sqrt{\kappa_r^2 + \kappa_z^2}}\ A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z)$$

where

for each value of κ_θ, $\kappa_{r:\theta} = \frac{j_{\kappa_\theta, \kappa_{r:\theta}}}{b}$ where
$\{j_{\kappa_\theta, \kappa_{r:\theta}}\}$ are the positive roots of $J_{\kappa_\theta}(j_{\kappa_\theta, \kappa_{r:\theta}}) = 0$

$$A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z) = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \int_a^b dr\ e^{-i\kappa_\theta\theta}\ \frac{1}{2\pi}\ e^{-i\kappa_z z}\ \frac{1}{2\pi}\ J_{\kappa_\theta}(\kappa_{r:\theta} r) r \times \frac{2}{b^2 {J_{\kappa_\theta+1}}^2(j_{\kappa_\theta, \kappa_{r:\theta}})} \times \frac{\partial\Psi(r, \theta, z; 0)}{\partial t}$$

$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$$\Psi_{steady\ state} = 0$$
$$\Psi_{transient} = \Psi_{transient\,\#1} + \Psi_{transient\,\#2}$$

$$\Psi_{transient\,\#1} = \sum_{\kappa_\theta=-\infty}^{\infty} \int_{-\infty}^{\infty} d\kappa_z \sum_{\kappa_{r:\theta}} e^{i\kappa_\theta \theta}\ e^{i\kappa_z z}\ [Y_{\kappa_\theta}(\kappa_{r:\theta}a)J_{\kappa_\theta}(\kappa_{r:\theta}r) - J_{\kappa_\theta}(\kappa_{r:\theta}a)Y_{\kappa_\theta}(\kappa_{r:\theta}r)] \times \cos\sqrt{\kappa_{r:\theta}^2 + \kappa_z^2}\,t\ A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z)$$

where

for each value of κ_θ, $\{\kappa_{r:\theta}\}$ are the positive roots of
$Y_{\kappa_\theta}(\kappa_{r:\theta}a)J_{\kappa_\theta}(\kappa_{r:\theta}b) - J_{\kappa_\theta}(\kappa_{r:\theta}a)Y_{\kappa_\theta}(\kappa_{r:\theta}b) = 0$
($J_{-\kappa_\theta}$ may replace Y_{κ_θ} if κ_θ is not an integer)

$$A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z) = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \int_a^b dr\ e^{-i\kappa_\theta \theta}\ \frac{1}{2\pi}\ e^{-i\kappa_z z}\ \frac{1}{2\pi} \times [Y_{\kappa_\theta}(\kappa_{r:\theta}a)J_{\kappa_\theta}(\kappa_{r:\theta}r) - J_{\kappa_\theta}(\kappa_{r:\theta}a)Y_{\kappa_\theta}(\kappa_{r:\theta}r)]r\ N_{\kappa_{r:\theta}} \times\ [\Psi(r,\theta,z;0) - \Psi_{steady\ state}]$$

$$N_{\kappa_{r:\theta}} = \frac{\pi^2}{2}\frac{\kappa_{r:\theta}^2 J_{\kappa_\theta}^2(\kappa_{r:\theta}b)}{J_{\kappa_\theta}^2(\kappa_{r:\theta}a) - J_{\kappa_\theta}^2(\kappa_{r:\theta}b)}$$

$$\Psi_{transient\,\#2} = \sum_{\kappa_\theta=-\infty}^{\infty} \int_{-\infty}^{\infty} d\kappa_z \sum_{\kappa_{r:\theta}} e^{i\kappa_\theta \theta}\ e^{i\kappa_z z}\ [Y_{\kappa_\theta}(\kappa_{r:\theta}a)J_{\kappa_\theta}(\kappa_{r:\theta}r) - J_{\kappa_\theta}(\kappa_{r:\theta}a)Y_{\kappa_\theta}(\kappa_{r:\theta}r)] \times \frac{\sin\sqrt{\kappa_{r:\theta}^2 + \kappa_z^2}\,t}{\sqrt{\kappa_r^2 + \kappa_z^2}}\ A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z)$$

where

for each value of κ_θ, $\{\kappa_{r:\theta}\}$ are the positive roots of
$Y_{\kappa_\theta}(\kappa_{r:\theta}a)J_{\kappa_\theta}(\kappa_{r:\theta}b) - J_{\kappa_\theta}(\kappa_{r:\theta}a)Y_{\kappa_\theta}(\kappa_{r:\theta}b) = 0$
($J_{-\kappa_\theta}$ may replace Y_{κ_θ} if κ_θ is not an integer)

$$A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z) = \int_0^{2\pi} d\theta \int_{-\infty}^{\infty} dz \int_a^b dr\ e^{-i\kappa_\theta \theta}\ \frac{1}{2\pi}\ e^{-i\kappa_z z}\ \frac{1}{2\pi} \times [Y_{\kappa_\theta}(\kappa_{r:\theta}a)J_{\kappa_\theta}(\kappa_{r:\theta}r) - J_{\kappa_\theta}(\kappa_{r:\theta}a)Y_{\kappa_\theta}(\kappa_{r:\theta}r)]r\ N_{\kappa_{r:\theta}}$$

$$\times \ \frac{\partial \Psi(r,\theta,z;0)}{\partial t}$$

$$N_{\kappa_{r:\theta}} = \frac{\pi^2}{2} \frac{\kappa_{r:\theta}^2 J_{\kappa_\theta}^2(\kappa_{r:\theta} b)}{J_{\kappa_\theta}^2(\kappa_{r:\theta} a) - J_{\kappa_\theta}^2(\kappa_{r:\theta} b)}$$

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$$\Psi = \Psi_{steady\ state} + \Psi_{transient}$$

$\Psi_{steady\ state} = 0$
$\Psi_{transient} = \Psi_{transient\,\#1} + \Psi_{transient\,\#2}$

$\Psi_{transient\,\#1} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{p=1}^{\infty} Y_{\ell m}(\phi,\theta)\ j_\ell(\kappa_{\ell,p} r)\ \cos\kappa_{\ell:r} t\ A_{\ell m}(\kappa_{\ell:r})$
where

$\kappa_{\ell,p} = s_{\ell,p}/b$ and$s_{\ell,p}$ is the pth positive zero of j_ℓ.
$A_{\ell m}(\kappa_{\ell:r}) = \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^{b} r^2\, dr\, Y^*_{\ell m}(\phi,\theta) \quad j_\ell(\kappa_{\ell,p} r)\ \frac{2}{b^3 j^2_{\ell+1}(s_{\ell,p})}$
$\times\ [\Psi(\theta,\phi,r;0) - \Psi_{steady\ state}]$

$\Psi_{transient\,\#2} = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{p=1}^{\infty} Y_{\ell m}(\phi,\theta)\ j_\ell(\kappa_{\ell,p} r)\ \frac{\sin\kappa_{\ell:r} t}{\kappa_{\ell:r}}\ A_{\ell m}(\kappa_{\ell:r})$
where

$\kappa_{\ell,p} = s_{\ell,p}/b$ and$s_{\ell,p}$ is the pth positive zero of j_ℓ.
$A_{\ell m}(\kappa_{\ell:r}) = \int_0^{\pi} \sin\phi\, d\phi \int_0^{2\pi} d\theta \int_0^{b} r^2\, dr\, Y^*_{\ell m}(\phi,\theta) \quad j_\ell(\kappa_{\ell,p} r)\ \frac{2}{b^3 j^2_{\ell+1}(s_{\ell,p})}$
$\times\ \frac{\partial\Psi(\theta,\phi,r;0)}{\partial t}$

$$\Psi = \Psi_1 + \Psi_2 + \Psi_3 + \Psi_4 + \Psi_5$$

$\Psi_1 = \sum_{\kappa_y} \sin \kappa_y y \;\; \eta_x(x;\kappa_y) \; A(\kappa_y)$
where
$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$
$\eta_x(x;\kappa_y) = \begin{cases} X - x & \text{if } \kappa_y = 0; \\ \sinh \kappa_y (X - x) & \text{otherwise.} \end{cases}$
$A(\kappa_y) = \int_0^Y \, dy \;\, \sin \kappa_y y \; \frac{2}{Y} \; M_{\kappa_y} \; f_{x=0}(y)$

$$M_{\kappa_y} = \begin{cases} \frac{1}{X} & \text{if } \kappa_y = 0; \\ \frac{1}{\sinh \kappa_y X} & \text{otherwise.} \end{cases}$$

$\Psi_2 = \sum_{\kappa_x} \sin \kappa_x x \;\; \eta_y(y;\kappa_x) \; A(\kappa_x)$
where
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\eta_y(y;\kappa_x) = \begin{cases} Y - y & \text{if } \kappa_x = 0; \\ \sinh \kappa_x (Y - y) & \text{otherwise.} \end{cases}$
$A(\kappa_x) = \int_0^X \, dx \;\, \sin \kappa_x x \; \frac{2}{X} \; M_{\kappa_x} \; f_{y=0}(x)$

$$M_{\kappa_x} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_x = 0; \\ \frac{1}{\sinh \kappa_x Y} & \text{otherwise.} \end{cases}$$

$\Psi_3 = \sum_{\kappa_y} \sin \kappa_y y \; \eta_x(x;\kappa_y) \; A(\kappa_y)$
where
$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$
$\eta_x(x;\kappa_y) = \begin{cases} x & \text{if } \kappa_y = 0; \\ \sinh \kappa_y x & \text{otherwise.} \end{cases}$
$A(\kappa_y) = \int_0^Y \, dy \;\, \sin \kappa_y y \; \frac{2}{Y} \; M_{\kappa_y} \; f_{x=X}(y)$

$$M_{\kappa_y} = \begin{cases} \frac{1}{X} & \text{if } \kappa_y = 0, \\ \frac{1}{\sinh \kappa_y X} & \text{otherwise.} \end{cases}$$

$\Psi_4 = \sum_{\kappa_x} \sin \kappa_x x \; \eta_y(y;\kappa_x) \; A(\kappa_x)$
where
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\eta_y(y;\kappa_x) = \begin{cases} y & \text{if } \kappa_x = 0; \\ \sinh \kappa_x y & \text{otherwise.} \end{cases}$
$A(\kappa_x) = \int_0^X \; dx \;\; \sin \kappa_x x \; \frac{2}{X} \; M_{\kappa_x} \; f_{y=Y}(x)$

$$M_{\kappa_x} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_x = 0, \\ \frac{1}{\sinh \kappa_x Y} & \text{otherwise.} \end{cases}$$

$\Psi_5 = \sum_{\kappa_x} \sum_{\kappa_y} \sin \kappa_x x \sin \kappa_y y \; A(\kappa_x, \kappa_y)$
where
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$
$A(\kappa_x, \kappa_y) = \int_0^X \; dx \int_0^Y \; dy \;\; \sin \kappa_x x \; \frac{2}{X} \;\; \sin \kappa_y y \; \frac{2}{Y} \; \frac{f_{interior}(x,y)}{\kappa_x^2 + \kappa_y^2}$

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$\Psi = \Psi_1$

$\Psi_1 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} \sum_{p=1}^{\infty} Y_{\ell m}(\phi, \theta)\ j_\ell(\kappa_{\ell,\,p} r)\ A_{\ell m}(\kappa_{\ell:r})$
$where$

$\kappa_{\ell,\,p} = s_{\ell,\,p}/b$ and$s_{\ell,\,p}$ is the pth positive zero of j_ℓ.
$A_{\ell m}(\kappa_{\ell:r}) = \int_0^{\pi}\ \sin\phi\, d\phi \int_0^{2\pi}\ d\theta \int_0^{b}\ r^2\, dr\ Y^*_{\ell m}(\phi, \theta) \quad j_\ell(\kappa_{\ell,\,p} r)\ \frac{2}{b^3 j^2_{\ell+1}(s_{\ell,p})}\ \frac{f_{interior}(\theta,\phi,r)}{\kappa^2_{\ell:r}}$

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$\Psi = \Psi_1 + \Psi_2$

$\Psi_1 = \ \eta_x(x;s)\ A(s)$
where
$\eta_x(x;s) = \begin{cases} x & \text{if } s = 0; \\ \sinh sx & \text{otherwise.} \end{cases}$
$A(s) = \ M_s\ F_{x=X}(s)$

$M_s = \begin{cases} \frac{1}{X} & \text{if } s = 0, \\ \frac{1}{\sinh sX} & \text{otherwise.} \end{cases}$

$\Psi_2 = \sum_{\kappa_x} \sin \kappa_x x\ A(\kappa_x;s)$
where
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$A(\kappa_x;s) = \int_0^X\ dx\ \ \sin \kappa_x x\ \frac{2}{X}$

$\times\ \frac{\Psi(x,t=0) + s\frac{\partial\Psi}{\partial t}(x,t=0)}{\kappa_x^2 + s^2}$

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$\Psi = \Psi_1$

$\Psi_1 = \ \eta_x(x;\omega)\ A(\omega)$
where
$$\eta_x(x;\omega) = \begin{cases} x & \text{if } \omega = 0; \\ \sin \omega x & \text{otherwise.} \end{cases}$$
$A(\omega) = \ M_\omega\ F_{x=X}(\omega)$

$$M_\omega = \begin{cases} \frac{1}{X} & \text{if } \omega = 0, \\ \frac{1}{\sin \omega X} & \text{otherwise.} \end{cases}$$

$$\Psi = \Psi_1$$

$\Psi_1 = \sum_{\kappa_x} \cos \kappa_x x \; A(\kappa_x; s)$
where
$\kappa_x = 0, \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$A(\kappa_x; s) = \int_0^X \; dx \; \cos \kappa_x x \; N_{\kappa_x} \; \frac{\Psi(x,t=0)}{\kappa_x^2 + s}$
$N_{\kappa_x} = \begin{cases} \frac{1}{X} & \text{if } \kappa_x = 0; \\ \frac{2}{X} & \text{otherwise} \end{cases}$

$$\Psi = \Psi_1 + \Psi_2$$

$\Psi_1 = \sum_{\kappa_\theta=-\infty}^{\infty} \sum_{\kappa_z} \; e^{i\kappa_\theta \theta} \sin \kappa_z z \; \eta_r(r, \kappa_\theta, \sqrt{\kappa_z^2 + s}) \; A(s; \kappa_\theta, \kappa_z)$
where

$\kappa_z = \frac{\pi}{Z}, \frac{2\pi}{Z}, \frac{3\pi}{Z}, \ldots$

$$\eta_r(r, \kappa_\theta, \sqrt{\kappa_z^2 + s}) = \begin{cases} 1 \text{ if } \kappa_\theta = \sqrt{\kappa_z^2 + s} = 0; \\ r^{\kappa_\theta} \text{ if } \kappa_\theta \neq 0 \text{ and } \sqrt{\kappa_z^2 + s} = 0; \text{ otherwise} \\ I_{\kappa_\theta}(\sqrt{\kappa_z^2 + s} r) \end{cases}$$

$A(s; \kappa_\theta, \kappa_z) = \int_0^{2\pi} \; d\theta \int_0^Z \; dz \; e^{-i\kappa_\theta \theta} \; \frac{1}{2\pi} \; \sin \kappa_z z \; \frac{2}{Z} \; N_r \; F_{r=b}(s; z, \theta)$

$$N_r = \begin{cases} 1 & \text{if } \kappa_\theta = \sqrt{\kappa_z^2 + s} = 0; \\ b^{-\kappa_\theta} & \text{if } \kappa_\theta \neq 0 \text{ and } \sqrt{\kappa_z^2 + s} = 0; \\ \frac{1}{I_{\kappa_\theta}(\sqrt{\kappa_z^2 + s} b)} & \text{if } \kappa_\theta \neq 0 \text{ and } \sqrt{\kappa_z^2 + s} \neq 0. \end{cases}$$

$\Psi_2 = \sum_{\kappa_\theta=-\infty}^{\infty} \sum_{\kappa_z} \sum_{\kappa_{r:\theta}} \; e^{i\kappa_\theta \theta} \sin \kappa_z z \; J_{\kappa_\theta}(\kappa_{r:\theta} r) \; A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z; s)$
where

$\kappa_z = \frac{\pi}{Z}, \frac{2\pi}{Z}, \frac{3\pi}{Z}, \ldots$
for each value of $\kappa_\theta, \kappa_{r:\theta} = \frac{j_{\kappa_\theta, \kappa_{r:\theta}}}{b}$ where
$\{j_{\kappa_\theta, \kappa_{r:\theta}}\}$ are the positive roots of $J_{\kappa_\theta}(j_{\kappa_\theta, \kappa_{r:\theta}}) = 0$

$$A(\kappa_{r:\theta}, \kappa_\theta, \kappa_z; s) = \int_0^{2\pi} \; d\theta \int_0^Z \; dz \int_a^b \; dr \; e^{-i\kappa_\theta \theta} \; \frac{1}{2\pi} \; \sin \kappa_z z \; \frac{2}{Z} \; J_{\kappa_\theta}(\kappa_{r:\theta} r) r \\ \times \frac{2}{b^2 J_{\kappa_\theta + 1}{}^2 (j_{\kappa_\theta, \kappa_{r:\theta}})} \\ \times \left(\frac{F_{interior}(r, \theta, z; s) + \Psi(r, \theta, z, t=0)}{\kappa_{r:\theta}^2 + \kappa_z^2 + s} \right)$$

$$\Psi = \Psi_1 + \Psi_2$$

$\Psi_1 = \sum_{\kappa_y} \sum_{\kappa_z} \cos \kappa_y y \cos \kappa_z z \ \cosh \sqrt{\kappa_y^2 + \kappa_z^2 + s^2} x \ A(s; \kappa_y, \kappa_z)$
where
$\kappa_y = 0, \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$
$\kappa_z = 0, \frac{\pi}{Z}, \frac{2\pi}{Z}, \frac{3\pi}{Z}, \ldots$

$$A(s; \kappa_y, \kappa_z) = \int_0^Y dy \int_0^Z dz \ \cos \kappa_y y \ N_{\kappa_y} \ \cos \kappa_z z \ N_{\kappa_z} \ M_{\sqrt{\kappa_y^2+\kappa_z^2+s^2}} \ F_{x=X}(s; y, z)$$

$$N_{\kappa_y} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_y = 0; \\ \frac{2}{Y} & \text{otherwise} \end{cases}$$

$$N_{\kappa_z} = \begin{cases} \frac{1}{Z} & \text{if } \kappa_z = 0; \\ \frac{2}{Z} & \text{otherwise} \end{cases}$$

$$M_{\sqrt{\kappa_y^2+\kappa_z^2+s^2}} = \begin{cases} 0 & \text{if } \sqrt{\kappa_y^2 + \kappa_z^2 + s^2} = 0; \\ \frac{1}{\sqrt{\kappa_y^2+\kappa_z^2+s^2} \sinh \sqrt{\kappa_y^2+\kappa_z^2+s^2} X} & \text{otherwise.} \end{cases}$$

$\Psi_2 = \sum_{\kappa_x} \sum_{\kappa_y} \sum_{\kappa_z} \cos \kappa_x x \cos \kappa_y y \cos \kappa_z z \ A(\kappa_x, \kappa_y, \kappa_z; s)$
where
$\kappa_x = 0, \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\kappa_y = 0, \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$
$\kappa_z = 0, \frac{\pi}{Z}, \frac{2\pi}{Z}, \frac{3\pi}{Z}, \ldots$

$$A(\kappa_x, \kappa_y, \kappa_z; s) = \int_0^X dx \int_0^Y dy \int_0^Z dz \ \cos \kappa_x x \ N_{\kappa_x} \ \cos \kappa_y y \ N_{\kappa_y} \ \cos \kappa_z z \ N_{\kappa_z} \times \frac{\Psi(x,y,z,t=0) + s \frac{\partial \Psi}{\partial t}(x,y,z,t=0)}{\kappa_x^2 + \kappa_y^2 + \kappa_z^2 + s^2}$$

$$N_{\kappa_x} = \begin{cases} \frac{1}{X} & \text{if } \kappa_x = 0; \\ \frac{2}{X} & \text{otherwise} \end{cases}$$

$$N_{\kappa_y} = \begin{cases} \frac{1}{Y} & \text{if } \kappa_y = 0; \\ \frac{2}{Y} & \text{otherwise} \end{cases}$$

$$N_{\kappa_z} = \begin{cases} \frac{1}{Z} & \text{if } \kappa_z = 0; \\ \frac{2}{Z} & \text{otherwise} \end{cases}$$

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$$\Psi = \Psi_1 + \Psi_2$$

$$\Psi_1 = \sum_{\kappa_x} \sum_{\kappa_y} \sin \kappa_x x \sin \kappa_y y \; e^{-\sqrt{\kappa_x^2+\kappa_y^2+s^2} z} \; A(s; \kappa_x, \kappa_y)$$

$where$

$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$

$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$

$$A(s; \kappa_x, \kappa_y) = \int_0^X \, dx \int_0^Y \, dy \;\; \sin \kappa_x x \; \frac{2}{X} \;\; \sin \kappa_y y \; \frac{2}{Y} \;\; F_{z=0}(s; x, y)$$

$$\Psi_2 = \sum_{\kappa_x} \sum_{\kappa_y} \int_0^\infty \, d\kappa_z \sin \kappa_x x \sin \kappa_y y \sin \kappa_z z \; A(\kappa_x, \kappa_y, \kappa_z; s)$$

$where$

$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$

$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$

$$A(\kappa_x, \kappa_y, \kappa_z; s) = \int_0^X \, dx \int_0^Y \, dy \int_0^\infty \, dz \;\; \sin \kappa_x x \; \frac{2}{X} \;\; \sin \kappa_y y \; \frac{2}{Y} \;\; \sin \kappa_z z \; \frac{2}{\pi} \times \frac{\Psi(x,y,z,t=0) + s \frac{\partial \Psi}{\partial t}(x,y,z,t=0)}{\kappa_x^2+\kappa_y^2+\kappa_z^2+s^2}$$

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$\Psi = \Psi_1$

$\Psi_1 = \sum_{\kappa_x} \sum_{\kappa_y} \sin \kappa_x x \sin \kappa_y y \;\; e^{-i\sqrt{\omega^2 - \kappa_x^2 - \kappa_y^2}\, z} \; A(\omega; \kappa_x, \kappa_y)$
$where$
$\kappa_x = \frac{\pi}{X}, \frac{2\pi}{X}, \frac{3\pi}{X}, \ldots$
$\kappa_y = \frac{\pi}{Y}, \frac{2\pi}{Y}, \frac{3\pi}{Y}, \ldots$

$A(\omega; \kappa_x, \kappa_y) = \int_0^X \; dx \int_0^Y \; dy \;\; \sin \kappa_x x \; \frac{2}{X} \;\; \sin \kappa_y y \; \frac{2}{Y} \;\; F_{z=0}(\omega; x, y)$

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$\Psi = \Psi_1$

$\Psi_1 = \sum_{\ell=0}^{\infty} \sum_{m=-\ell}^{\ell} Y_{\ell m}(\phi, \theta)\ h_\ell^{(2)}(\omega r)\ A(\omega; \ell, m)$
where

$A(\omega; \ell, m) = \int_0^{\pi}\ \sin\phi\, d\phi \int_0^{2\pi}\ d\theta\ Y_{\ell m}^{*}(\phi, \theta) \quad \frac{1}{h_\ell^{(2)}(\omega a)}\ F_{r=a}(\omega; \theta, \phi)$